Recall:

Poisson's eq. $\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$

(WF) Find $u \in H_0^1$ s.t. $a(u,v) = \ell(v) \quad \forall v \in H_0^1$

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \quad , \quad \ell(v) = \int_\Omega f v$$

(FE) Find $u_h \in V_h^0$ s.t. $a(u_h, \chi) = \ell(\chi) \quad \forall \chi \in V_h^0$

linear system $\quad S \xi = b$

Assembly of stiffness matrix S, load vector b.

Error estimate in energy norm

$$\|u - u_h\|_{L^2} \underset{P}{\leq} C \, \|u - u_h\|_E \leq C \cdot h \, \|u\|_{H^2} \quad , \quad \|v\|_E = \sqrt{a(v,v)}$$

Poincaré

Chapter XII: FEM for heat equations in higher dimensions

Goal: Provide some stability estimates.
Present FE discretisation of heat eq. in dimension n. Error estimates.

1) Solutions to heat equations:

(Informal discussion)

- The sol. to the linear ODE

$$\begin{cases} \dot{u}(t) = a \cdot u(t) + f(t) \\ u(0) = u_0, \end{cases}$$

where $u_0, a \in \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ are given,

reads:

$$u(t) = e^{at} u_0 + \int_0^t e^{a(t-s)} f(s) ds \quad \text{(chapter VII)}$$

It is called variation of constants / Duhamel

Rem. • We observe that the term

$e^{at} u_0$ is the sol. to

$$\dot{u}(t) = a u(t), \; u(0) = u_0.$$

- The second term, $\int_0^t e^{a(t-s)} f(s)\,ds$, can be seen as a convolution.

- The above, in particular Duhamel, can be extended to linear systems of ODE:

$$\begin{cases} \dot{u}(t) = Au(t) + f(t) \\ u(0) = u_0, \end{cases}$$

where A is an $n \times n$ matrix, $f(t), u_0, u$ are vectors.

The sol. reads

$$u(t) = e^{At} u_0 + \int_0^t e^{A(t-s)} f(s)\,ds,$$

where

$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}$$

- Under suitable assumptions, Duhamel's formula can be extended to bounded operator A on a Hilbert space and f taking values in this Hilbert space.

<u>Illustration:</u> Heat equation in $\mathbb{R}^d$:

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + f(x,t) & x \in \mathbb{R}^d,\ t \in \mathbb{R}_+ \\ u(x,0) = u_0(x) & x \in \mathbb{R}^d \end{cases}$$

Denote Laplacian $A := \Delta$, and set

$$u_t = Au + f$$

Then, Duhamel's formula reads

$$u(x,t) = e^{At} u_0(x) + \int_0^t e^{A(t-s)} f(x,s)\,ds$$

or

$$(*) \quad u(x,t) = E(t)\,u_0(x) + \int_0^t E(t-s)\, f(x,s)\,ds,$$

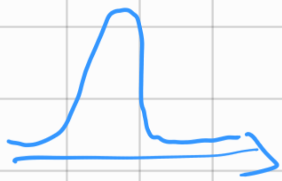
where $E(t)$ is the solution operator of DE

$u_t = Au$ or $u_t = \Delta u$ (recall: $e^{At} = e^{\Delta t}$)

<u>Rem:</u> The eq. (*) can also be written as

$$u(x,t) = \big(G(\cdot,t) * u_0\big)(x) + \int_0^t \int_{\mathbb{R}^d} G(x-y,t-s)\, f(y,s)\,dy\,ds,$$

where $G(x,t) = (4\pi t)^{-d/2}\, e^{-|x|^2/(4t)}$



is called Gauss Kernel / Green's function for heat

- The above discussion can be extended to

$$\begin{cases} u_t - \Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u(\cdot, 0) = u_0 & \text{in } \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^d$ and $\partial\Omega$ are nice.

Duhamel's formula reads

$$u(t) = E(t)\, u_0 + \int_0^t E(t-s) f(s)\, ds,$$

we write $u(t)$ for $u(x,t)$ and similarly for f.

[Rem: This $E(t)$ needs to be defined, not same as above.]

Since one has $\|E(t)v\|_{L^2} \leq C \cdot \|v\|_{L^2}$ $\forall t, \forall v \in L^2$,

one gets stability estimates

$$\|u(t)\|_{L^2} \leq \|u_0\|_{L^2} + \int_0^t \|f(s)\|_{L^2}\, ds$$

Other estimates $\rightsquigarrow$ book p. 263 (if interested)

2) Variational formulation and FE approximation:

Consider

$$\begin{cases} u_t - \Delta u = f & \text{in } \Omega \times \mathbb{R}_+ \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}_+ \\ u(\cdot, 0) = u_0 & \text{in } \Omega \,, \end{cases}$$

where $\Omega \subset \mathbb{R}^d$ and $\partial\Omega$ are nice.

- To get the variational formulation, we test against $v \in H_0^1(\Omega)$ and use Green's formula:

$$\int_\Omega u_t v \, dx - \underbrace{\int_\Omega \Delta u v \, dx}_{\text{Green}} = \int_\Omega f v \, dx$$

$$\underbrace{\int_{\partial\Omega} (\nabla u \cdot n) v \, ds}_{= 0 \text{ since } v = 0 \text{ on } \partial\Omega} - \int_\Omega \nabla u \cdot \nabla v \, dx$$

We get here (VF):

(VF) Find $u(\cdot, t) \in H_0^1$ for $0 < t \leq T$, s.t.

$$\begin{cases} (u_t(\cdot, t), v)_{L^2} + (\nabla u(\cdot, t), \nabla v)_{L^2} = (f(\cdot, t), v)_{L^2} \quad \forall v \in H_0^1 \\ u(x, 0) = u_0(x) \qquad \forall x \in \Omega \end{cases}$$

• To get the FE problem, we consider a triangulation T_h of Ω and the space

$$V_h^0(\Omega) = \{ v : \Omega \to \mathbb{R} ; \text{ cont. pw linear on } T_h, \ v|_{\partial\Omega} = 0 \}$$

$$= \operatorname{span}(\{\varphi_j\}_{j=1}^{n_i}), \text{ where } n_i \text{ denotes}$$

the number of interior nodes

This gives us FE problem:

(FE) Find $u_h(\cdot, t) \in V_h^0(\Omega)$ s.t.

$$(u_{h,t}(\cdot,t), \chi)_{L^2} + (\nabla u_h(\cdot,t), \nabla \chi)_{L^2} = (f(\cdot,t), \chi)_{L^2}$$

$$u_h(\cdot, 0) = \pi_h u_0 \qquad \forall \chi \in V_h^0$$

• Writing $u_h(x,t) = \sum_{j=1}^{n_i} \xi_j(t)\varphi_j(x)$ and taking

$\chi = \varphi_i$ for $i = 1, 2, \dots, n_i$, one gets a linear

system of ODE:

$$\Big(M\dot{\xi}(t) + S\xi(t) = F(t)$$

$$\big\{ \; \xi(0) = \xi_0 ,$$

where $M \rightsquigarrow$ mass matrix $(n_i \times n_i)$

$S \rightsquigarrow$ stiffness matrix $(n_i \times n_i)$

$F(t) \rightsquigarrow$ "load vector" $(1 \times n_i)$

$\xi(t) \rightsquigarrow$ unknown vector $(1 \times n_i)$

3) Semi-discrete error estimates:

Recall:

(VF) Find u s.t. $(u_t, v) + \underbrace{(\nabla u, \nabla v)}_{a(u,v)} = \underbrace{(f, v)}_{\ell(v)}$

$u(\cdot, 0) = u_0$

Stability estimate $\|u(t)\|_{L^2} \leq \|u_0\|_{L^2} + \int_0^t \|f(s)\|_{L^2} \, ds$

(FE) Find u_h s.t. $\begin{cases} (u_{h,t}, \chi) + (\nabla u_h, \nabla \chi) = (f, \chi) \\ u_h(\cdot, 0) = \Pi_h u_0 \end{cases}$

Stability estimate $\|u_h(t)\|_{L^2} \leq \|\Pi_h u_0\|_{L^2} + \int_0^t \|f(s)\|_{L^2} \, ds$

Interpolation error : $\|v - \Pi_h v\|_{L^2(\Omega)} \leq C \cdot h^2 \|v\|_{H^2(\Omega)}$

Next, we define :

Def: Define $R_h : H_0^1(\Omega) \to \mathring{V}_h(\Omega)$ the orthogonal projection with respect to the energy inner product

For $v \in H_0^1(\Omega)$: $a(R_h v - v, \chi) = 0 \quad \forall \chi \in \mathring{V}_h(\Omega)$,

where $a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx$.

The operator R_h is called Ritz projection (or elliptic projection)

We have the following result:

Th: Let Ω be convex and bounded. For $s = 1, 2$, one has:

$$\| R_h v - v \|_{L^2(\Omega)} \leq C \cdot h^s \| v \|_{H^s(\Omega)}$$

$$\forall \, v \in H^s(\Omega) \cap H_0^1(\Omega).$$

All the above can be used to estimate the error of FE approximation u_h!

Th: Let u be sol. to (VF), let u_h be sol. (FE),

assume that both are nice enough.

One has the error estimate:

$$\|u_h(\cdot,t) - u(\cdot,t)\|_{L^2(\Omega)} \leq \|\Pi_h u_0 - u_0\|_{L^2(\Omega)} + $$

$$+ C \cdot h^2 \cdot \left(\|u_0\|_{H^2(\Omega)} + \int_0^t \|u_t(\cdot,s)\|_{L^2} ds \right)$$

$$\approx C \cdot h^2$$