Recall:

- Heat eq. $\begin{cases} u_t - \Delta u = f & \text{in } \Omega \subset \mathbb{R}^d \quad (d=2,3 \text{ f.ex}) \\ u = 0 & \text{on } \partial\Omega \\ u(\cdot,0) = u_0 & \text{in } \Omega \end{cases}$

- (VF) Find $u(\cdot,t) \in H_0^1(\Omega)$ s.t.

$$(u_t, v) + \underbrace{a(u,v)}_{= (\nabla u, \nabla v) = \int_\Omega \nabla u \cdot \nabla v \, dx} = (f, v) \qquad \forall\, v \in H_0^1(\Omega)$$

stability: $\|u(\cdot,t)\|_{L^2} \leq \|u_0\|_{L^2} + \int_0^t \|f(\cdot,s)\|_{L^2} \, ds$

- (FE) Find $u_h(\cdot,t) \in V_h^0$ s.t.

$$(u_{h,t}, \chi) + a(u_h, \chi) = (f, \chi) \qquad \forall\, \chi \in V_h^0$$

- Ritz projection $R_h : H_0^1 \to V_h^0$: $a(R_h v - v, \chi) = 0 \;\; \forall\, \chi \in V_h^0$

$\Updownarrow$

$a(R_h v, \chi) = a(v, \chi)$

$$\|R_h v - v\|_{L^2} \leq C \cdot h^2 \, \|v\|_2$$

$\to$ H^2-norm

Exam: MVE455, canvas.

Th: Let u be sol. to (VF) (heat eq.) and u_h sol. (FE) (cG(1) approx). Assume f, Ω, u, u_h, u_0 nice.

One has the error estimate:

$$\|u_h(\cdot,t) - u(\cdot,t)\|_{L^2(\Omega)} \leq \underbrace{\|\pi_h u_0 - u_0\|_{L^2(\Omega)}}_{\approx C \cdot h^2} + C \cdot h^2 \left(\|u_0\|_{H^2} + \int_0^t \|u_t(\cdot,s)\|_{H^2} ds \right)$$

Proof:

- Write the error $u_h - u$ as

$$u(t) - u_h(t) = \underbrace{u(t) - R_h u(t)}_{err_1(t)} + \underbrace{R_h u(t) - u_h(t)}_{err_2(t)}$$

- For the first term, we get:

$$\|err_1(t)\|_{L^2} \underset{\substack{\uparrow \\ \text{Def } err_1}}{=} \|u(t) - R_h u(t)\|_{L^2} \underset{\substack{\uparrow \\ \text{error of Ritz, see above}}}{\leq} C \cdot h^2 \|u(t)\|_{H^2}$$

- For the second term, err_2, we use a kind of variational formulation: For $\varphi \in V_h^0$ consider:

$$((err_2)_t, \varphi) + a(err_2, \varphi) \underset{\substack{\uparrow \\ \text{Def of } err_2}}{=} ((R_h u - u_h)_t, \varphi) + a(R_h u - u_h, \varphi) =$$

$$\underset{\text{linearity}}{=} ((R_h u)_t, \varphi) + a(R_h u, \varphi) \underbrace{-(u_{h,t}, \varphi) - a(u_h, \varphi)}_{-(f,\varphi) \text{ since } u_h \text{ solves (FE)}}$$

$$= ((R_h u)_t, \varphi) + \underbrace{a(R_h u, \varphi)}_{a(u,\varphi) \text{ Def. of Ritz}} - (f, \varphi) =$$

$$= ((R_h u)_t, \varphi) + \underbrace{a(u,\varphi) - (f,\varphi)}_{-(u_t,\varphi) \text{ since } u \text{ sol. to (VF)}} =$$

$$\underset{\text{linearity}}{=} ((R_h u - u)_t, \varphi) \underset{\text{Def err}_1}{=} -((\text{err}_1)_t, \varphi)$$

This gives us

$$((\text{err}_2)_t, \varphi) + a(\text{err}_2, \varphi) = -((\text{err}_1)_t, \varphi) \quad \forall \varphi \in V_h^0$$

Using stability estimate for heat eq, we end up with:

$$\|\text{err}_2(t)\|_{L^2} \leq \|\text{err}_2(0)\|_{L^2} + \int_0^t \|(\text{err}_1)_t(s)\|_{L^2} \, ds$$

Next, we estimate these 2 terms:

$$\|err_2(0)\|_{L^2} \underset{\text{Def } err_2}{=} \| R_h u(0) - u_0 + u_0 - u_h(0) \|_{L^2} \overset{\Delta}{\leq}$$

$$\underset{\text{Def } u_0,\ u_h(0)}{\leq} \|R_h u_0 - u_0\|_{L^2} + \|u_0 - \Pi_h u_0\|_{L^2} \leq$$

$$\leq C \cdot h^2 \|u_0\|_{H^2} + \|u_0 - \Pi_h u_0\|_{L^2}$$

and

$$\|(err_1)_t(s)\|_{L^2} \underset{\text{Def } err_1}{=} \|(u - R_h u)_t(s)\|_{L^2} \underset{\text{"}R_h \leftrightarrow t\text{"}}{=} \|(u_t - R_h(u_t))(s)\|_{L^2} \underset{\text{error Ritz}}{\leq}$$

$$\leq C h^2 \|u_t(s)\|_{H^2}$$

- Finally, we collect all these estimates and get

$$\|u(t) - u_h(t)\|_{L^2} \leq \|err_1(t)\|_{L^2} + \|err_2(t)\|_{L^2} \leq$$

$$\leq C h^2 \underbrace{\|u(t)\|_{H^2}}_{u(t) = u_0 + \int_0^t u_t(s)\,ds} + C h^2 \|u_0\|_{H^2} + \|u_0 - \Pi_h u_0\|_{L^2} + \overset{Ch^2}{\downarrow}\int_0^t \|u_t(s)\|_{H^2}\,ds$$

$$\leq \|u_0 - \Pi_h u_0\|_{L^2} + C h^2 \|u_0\|_{H^2} + C h^2 \int_0^t \|u_t(s)\|_{H^2}\,ds \qquad \blacksquare$$

Chapter XIII: FEM for wave equations in $\mathbb{R}^d$

Goal: Analyse the exact sol., present and analyse

cG(1) approximation.

1) Conservation of energy:

Consider

$$(W)\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \Omega \times \mathbb{R}_{\geq 0} \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}_{\geq 0} \\ u(\cdot, 0) = u_0 & \text{in } \Omega \\ u_t(\cdot, 0) = v_0 & \text{in } \Omega, \end{cases}$$

$\Omega \subset \mathbb{R}^d$ $(d = 2,3)$.

We multiply (W) with u_t and integrate $\int_\Omega$ to get:

$$\int_\Omega \underbrace{u_{tt}(x,t)\, u_t(x,t)}_{\frac{1}{2}\frac{d}{dt}(u_t)^2}\, dx - \underbrace{\int_\Omega \Delta u(x,t)\, u_t(x,t)\, dx}_{\text{Green's formula}} = 0$$

$$0 - \int_\Omega \underbrace{\nabla u(x,t) \nabla u_t(x,t)}_{\frac{1}{2}\frac{d}{dt}(\nabla u)^2}\, dx$$

This provides the following equality:

$$\frac{1}{2}\frac{d}{dt}\left(\|u_t(\cdot,t)\|_{L^2}^2 + \|\nabla u(\cdot,t)\|_{L^2}^2\right) = 0$$

Hence, we get conservation of energy:

$$\frac{1}{2}\|u_t(\cdot,t)\|_{L^2}^2 + \frac{1}{2}\|\nabla u(\cdot,t)\|_{L^2}^2 = \frac{1}{2}\|v_0\|_{L^2}^2 + \frac{1}{2}\|\nabla u_0\|_{L^2}^2 = \text{Const.}$$

$$\forall t \geq 0.$$

2) Variational formulation and FE problem:

Consider

$$(PDE)\begin{cases} u_{tt} - \Delta u = f & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega \\ u(\cdot,0) = u_0,\ u_t(\cdot,0) = v_0 & \text{in } \Omega \end{cases} \qquad (\text{for } t \geq 0)$$

Again, we test against $v \in H_0^1(\Omega)$, integrate $\int_\Omega$, use Green to get the weak formulation for (PDE):

(VF)
Find $u(\cdot,t) \in H_0^1(\Omega)$ s.t. $(u_{tt}(\cdot,t), v) + \underbrace{a(u(\cdot,t),v)}_{= \int_\Omega \nabla u(\cdot,t)\cdot\nabla v(\cdot)} = (f(\cdot,t),v)\ \forall v \in H_0^1$

To get a FE problem, consider a triangulation/mesh T_h of Ω and the corresponding space V_h^0 and get the problem:

(FE) Find $u_h(\cdot,t) \in V_h^0$ s.t.

$$\begin{cases} (u_{h,tt}(\cdot,t), \chi) + a(u_h(\cdot,t), \chi) = (f(\cdot,t), \chi) \quad \forall \chi \in V_h^0 \\ u_h(0) = \Pi_h u_0 \\ u_{t,h}(0) = \Pi_h v_0 \end{cases}$$

Finally, writing $u_h(x,t) = \sum_{j=1}^{n_i} \xi_j(t) \varphi_j(x)$ and taking $\chi(x) = \varphi_i(x)$ for $i = 1, 2, \dots, n_i$, we get the linear system of ODE:

$$\begin{cases} M \cdot \ddot{\xi}(t) + S' \xi(t) = F(t) \\ \xi(0) = \xi_0 \\ \dot{\xi}(0) = \eta_0 \end{cases},$$

with the usual suspects for $M, S', F \dots$

3) <u>A priori error estimates for the semi-discrete problem (FE prob.)</u>:

<u>Th</u>: Let u be sol. (VF) (wave eq.) and u_h its FE approximation ($cG(1)$). Assume, $u, u_h, \Omega, \partial\Omega, f, u_0, v_0$ nice enough. One has

The estimate: For $t \geq 0$:

$$\|u_h(t) - u(t)\|_{L^2(\Omega)} \leq C \cdot \left(|\Pi_h u_0 - R_h u_0|_1 + \|\Pi_h v_0 - R_h v_0\|_{L^2} \right)$$

$$+ C h^2 \left(\|u(t)\|_{H^2} + \int_0^t \|u_{tt}(s)\|_{H^2} \, ds \right),$$

$\approx C \cdot h^2$

where R_h Ritz projection and semi-norm

$$|w|_1 = \left(\sum_{|\alpha|=1} \|D^\alpha w\|_{L^2}^2 \right)^{1/2} \qquad \left(\text{"} |w|_1 = \|\nabla w\|_{L^2} \text{"} \right)$$

[Book p. 274 for more estimates if interested]

Proof:

- We start as in the proof of error estimate for the heat eq. and write the error as

$$u_h(t) - u(t) = \underbrace{u_h(t) - R_h u(t)}_{\theta(t)} + \underbrace{R_h u(t) - u(t)}_{\varrho(t)} = \theta(t) + \varrho(t).$$

- For the term $\varrho(t)$, we get (as before)

$$\|\varrho(t)\|_{L^2} \leq C \cdot h^2 \cdot \|u(t)\|_{H^2} \quad \text{(error of Ritz)}$$

Similarly, we also get

$\|\rho_t(t)\|_{L^2} \leq C h^2 \|u_t(t)\|_{H^2}$ and

$\|\rho_{tt}(t)\|_{L^2} \leq C \cdot h^2 \|u_{tt}(t)\|_{H^2}$.

- For the error $\theta(t) = u_h(t) - R_h u(t)$, we get the following equation:

$$(\theta_{tt}(t), \chi) + (\nabla\theta(t), \nabla\chi) = -(\rho_{tt}(t), \chi) \quad \forall \chi \in V_h^\circ$$

Taking $\chi = \theta_t(t) \in V_h^\circ$ above, gives us:

$$\underbrace{(\theta_{tt}(t), \theta_t(t)) + (\nabla\theta(t), \nabla\theta_t(t))}_{\frac{1}{2}\frac{d}{dt}\left(\|\theta_t(t)\|_{L^2}^2 + \underbrace{\|\nabla\theta(t)\|_{L^2}^2}_{|\theta(t)|_1^2}\right)} = -(\rho_{tt}(t), \theta_t(t))$$

(see cons. energy)

(def semi-norm)

For the RHS, we use C-S and get

$$\frac{1}{2}\frac{d}{dt}\left(\|\theta_t(t)\|_{L^2}^2 + |\theta(t)|_1^2\right) \leq \|\rho_{tt}(t)\| \cdot \|\theta(t)\|_{L^2}$$

Integrating over time, $\int_0^t \ldots ds$, we get

$$\frac{1}{2}\left(\|\theta_t(t)\|_{L^2}^2 + |\theta(t)|_1^2 \right) \leq \frac{1}{2}\left(\|\theta_t(0)\|_{L^2}^2 + |\theta(0)|_1^2 \right) +$$

$$+ \int_0^t \|\rho_{tt}(s)\|_{L^2} \cdot \underbrace{\|\theta(s)\|_{L^2}}_{\leq \max\limits_{0 \leq s \leq t} \|\theta(s)\|_{L^2}} ds$$

This gives us:

$$\|\theta_t(t)\|_{L^2}^2 + |\theta(t)|_1^2 \leq \|\theta_t(0)\|_{L^2}^2 + |\theta(0)|_1^2 +$$

$$+ 2 \int_0^t \|\rho_{tt}(s)\|_{L^2} ds \cdot \max_{0 \leq s \leq t} \|\theta(s)\|_{L^2}$$