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Chapter 12: FEM for heat equations in higher dimensions (summary)

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Goal: Study the exact solution to heat equations (stability) and derive a FE discretisation for these PDEs.

- Let $\Omega \subset \mathbb{R}^d$ be a nice domain with smooth boundary. The solution to the inhomogeneous heat equation

$$\begin{cases} u_t - \Delta u = f & \text{in} \quad \Omega \times \mathbb{R}_+ \\ u = 0 & \text{on} \quad \partial\Omega \times \mathbb{R}_+ \\ u(\cdot, 0) = u_0 & \text{in} \Omega \end{cases}$$

  may be expressed thanks to Duhamel's formula/the variation of constants formula

$$u(t) = E(t)u_0 + \int_0^t E(t-s)f(s)\,\mathrm{d}s,$$

  where one writes $u(t)$ for $u(\cdot, t)$ and similarly for $f(s)$. Here, $E(t) = \mathrm{e}^{\Delta t}$ denotes the solution operator to the linear part $u_t - \Delta u = 0$.

- Since $\|E(t)v\|_{L^2(\Omega)} \leq \|v\|_{L^2(\Omega)}$ for all $t > 0$, one directly gets the stability estimates

$$\|u(t)\|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega)} + \int_0^t \left\|f(s)\right\|_{L^2(\Omega)} \mathrm{d}s.$$

  In addition, when $f \equiv 0$ (see page 263 in the book if interested in details), one has the following estimate

$$\int_0^t \|\nabla u(\cdot, s)\|^2_{L^2(\Omega)} \,\mathrm{d}s \leq \frac{1}{2} \|u_0\|_{L^2(\Omega)}.$$

- The variational formulation of the above heat equation reads:

  Find $\quad u(\cdot, t) \in H^1_0(\Omega)$, for $\quad t > 0, \quad$ such that $\quad (u_t, v)_{L^2(\Omega)} + (\nabla u, \nabla v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)} \quad \forall v \in H^1_0(\Omega)$

  and $u(\cdot, 0) = u_0$ in Ω for the initial value. We also denote $a(u, v) = (\nabla u, \nabla v)_{L^2(\Omega)}$.

- Let now T_h denote a mesh of Ω and V_h the space of continuous piecewise linear functions of T_h. Consider the space $V_h^0 = \{v \colon \Omega \to \mathbb{R} : \ v \quad \text{continuous pw linear on } T_h \text{ and } v = 0 \text{ on } \partial\Omega\}$ and observe that $V_h^0 = \text{span}(\{\varphi_j\}_{j=1}^{n_i})$, where n_i denotes the number of interior nodes. The finite element problem for the above heat equation reads

  Find $\quad u_h(\cdot, t) \in V_h^0(\Omega)$, for $\quad t > 0, \quad$ such that $\quad (u_{h,t}, \chi)_{L^2(\Omega)} + (\nabla u_h, \nabla \chi)_{L^2(\Omega)} = (f, \chi)_{L^2(\Omega)} \quad \forall \chi \in V_h^0(\Omega)$

  and $u_h(x, 0) = \pi_h u_0(x)$ in Ω for the initial value.

  As always, writing $u_h(x, t) = \sum_{j=1}^{n_i} \zeta_j(t)\varphi_j(x)$ and taking $\chi = \varphi_i$ in the FE gives the system of linear ODEs

$$\begin{cases} M\dot{\zeta}(t) + S\zeta(t) = F(t) \\ \zeta(0) = \zeta_0. \end{cases}$$

  One has the following a priori error estimate for the FE approximation of the heat equation

$$\|u_h(\cdot, t) - u(\cdot, t)\|_{L^2(\Omega)} \leq \|\pi_h u_0 - u_0\|_{L^2(\Omega)} + Ch^2 \left(\|u_0\|_{H^2(\Omega)} + \int_0^t \|u_t(\cdot, s)\|_{H^2(\Omega)} \,\mathrm{d}s \right),$$

  where we recall that $\pi_h u_0$ denotes the continuous pw linear interpolant of u_0.

- A useful tool in the proof of the above result, and in general, is the Ritz projection $R_h\colon H_0^1(\Omega) \to V_h^0(\Omega)$. This is defined as the orthogonal projection with respect to the energy inner product: For $v \in H_0^1(\Omega)$ one has
$$a(R_h v - v, \chi) = 0 \quad \forall \chi \in V_h^0.$$
Under some assumptions on the domain Ω and v, one has the estimate
$$\|R_h v - v\|_{L^2(\Omega)} \leq Ch^2 \|v\|_{H^2(\Omega)}.$$

Further resources:

- wikipedia.org
- pims.math.ca
- wikiversity.org
- wikiversity.org
- wikiversity.org
- fenicsproject.org
- math.uci.edu