Mathematics Chalmers & GU

MVE455: Partial Differential Equations, 2020–06–09, 14:00-18:00

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An open book exam.
Each problem gives max 6p. Valid bonus points will be added to the scores.
Breakings **3**: 15-21p, **4**: 22-28p, **5**: 29p-

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1. Prove that if f has square integrable second derivatives, then its L_2-projection $P_h f$ satisfies

$$\|f - P_h f\|^2 \le C \sum_{K\in\mathcal{T}_h} h_K^4 \|D^2 f\|_K^2.$$

2. A model problem for the traffic flow of cars with speed u and density ρ can be written as

$$\dot\rho + (u\rho') = 0. \tag{1}$$

Assume that $u = c - \varepsilon(\rho'/\rho)$ (**), and derive convection-diffusion equation $\dot\rho + c\rho' - \varepsilon\rho'' = 0$. Give a full motivation for this choice (**) of u.

3. Determine the two point boundary value problem having FEM linear system of equations, viz.

$$S\xi = \mathbf{a} + \mathbf{b},$$

where S is $(m+1)\times(m+1)$ matrix with EVEN m and with $s_{ii} = 2/h$; $i = 1,\dots m$, and $s_{m+1,m+1} = 1/h$, $s_{i,i+1} = s_{i+1,i} = -1/h$, $i = 1,\dots m$. Further $\mathbf{a}$, and $\mathbf{b}$ are $(m+1)\times 1$ vectors:

$$a_i = 0, \quad i = 1,2,\dots m, \qquad a_{m+1} = 1$$

$$b_i = 0, \quad i = 1,\dots,\frac{m}{2}+1, \qquad b_i = -3h, \quad i = \frac{m}{2}+2,\dots,m+1.$$

4. Formulate and prove the Lax-Milgram theorem in full details for the $2d$-problem

$$-\Delta u + \alpha u' = f, \quad x\in\Omega, \qquad n\cdot\nabla u = 0, \quad x\in\partial\Omega.$$

5. a) Consider the Schrödinger equation

$$i\,\dot u - \Delta u = 0, \quad in\ \Omega, \qquad u = 0, \quad on\ \partial\Omega,$$

where $i = \sqrt{-1}$ and $u = u_1 + u_2$. Show that the total probability: $\|u\|_{L_2(\Omega)}$ is time independent.
b) Consider the corresponding eigenvalue problem:

$$-\Delta u = \lambda u, \quad in\ \Omega, \qquad u = 0, \quad on\ \partial\Omega.$$

Show that for the eigenpair (λ, u), $\lambda > 0$. Give the relation between $\|u\|$ and $\|\nabla u\|$.

6. Consider the problem

$$-u'' + xu' + u = f \quad \text{in } I = (0,1), \qquad u(0) = u'(1) = 0,$$

where ε in a positive constant, and $f \in L_2(I)$. Prove the following L_2-stability:

$$\|u''\| \le \|f\|.$$

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