Mathematics Chalmers & GU

TMA372/MMG800: Partial Differential Equations, 2020–03–16, 8:30-12:30

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Calculators, formula notes and other subject related material are not allowed.
Each problem gives max 5p. Valid bonus poits will be added to the scores.
Breakings for **Chalmers; 3**: 15-21p, **4**: 22-28p, **5**: 29p-, and for **GU; G**: 15-26p, **VG**: 27p-

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1. Prove that for $0 < b - a \le 1$,

$$\|f\|_{L_1(a,b)} \le \|f\|_{L_2(a,b)} \le \|f\|_{L_\infty(a,b)}.$$

2. Show that there is a natural energy, associated with the solution u of the equation below, which is preserved in time (as t increases) for $b = 0$, and decreases as time increases for $b > 0$.

$$\ddot{u} + b\dot{u} - u'' = 0 \quad 0 < x < 1, \qquad u(0,t) = u(1,t) = 0, \quad (b > 0).$$

3. Solution of problem: $-(au')' = f$, $0 < x < 1$, $u(0) = u'(1) = 0$, minimizes total energy: $F(v) = \frac{1}{2}\int_0^1 a(v')^2 - \int_0^1 fv$, i.e., $F(u) = \min_{v\in V} F(v); u \in V$ where V is some function space.
(a) Show that for corresponding discrete minimum: $F(U) = \min_{v\in V_h} F(v)$, $U \in V_h \subset V$, $(a = 1)$:

$$F(U) = F(u) + \frac{1}{2}||(u-U)'||^2, \qquad a \equiv 1.$$

(b) Let $a = 1$ and show an a posteriori error estimate for the discrete energy minimum: i.e., for $|F(U) - F(u)|$. Note that V_h is the space of piecewise linear functions on subintervals of length h.

4. Let $\mathbf{n}$ be the outward unit normal to $\Gamma := \partial\Omega$ and consider the boundary value problem

$$-\Delta u + u = f, \quad \text{in } \Omega \subset \mathbb{R}^d, \ \ \mathbf{n}\cdot\nabla u = g, \ \text{on } \Gamma := \partial\Omega.$$

(a) Show the following stability estimate: for some constant C,

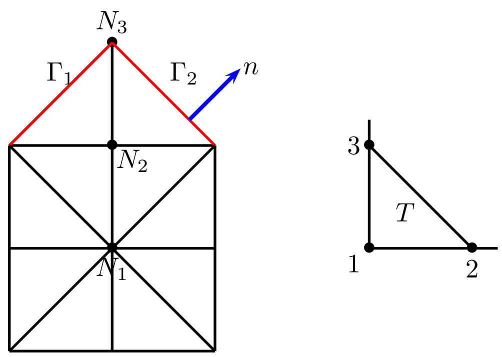
$$||\nabla u||^2_{L_2(\Omega)} + ||u||^2_{L_2(\Omega)} \le C(||f||^2_{L_2(\Omega)} + ||g||^2_{L_2(\Gamma)}).$$

(b) Formulate a finite element method for the $1D$-case and derive the resulting system of equations for $\Omega = [0,1]$, $f(x) = 1$, $g(0) = 3$ and $g(1) = 0$.

5. Compute the stiffness and mass matrices as well as load vector for the cG(1) approximation for

$$-\varepsilon\Delta u + u = 3, \quad x \in \Omega; \qquad u = 0, \quad x \in \partial\Omega \setminus (\Gamma_1 \cup \Gamma_2), \quad \nabla u \cdot \mathbf{n} = 0, \quad x \in \Gamma_1 \cup \Gamma_2,$$

where $\varepsilon > 0$ and $\mathbf{n}$ is the outward unit normal to $\partial\Omega$, (obs! 3 nodes N_1, N_2 and N_3, see Fig.)



6. Formulate and prove the Lax-Milgram Theorem (same version as in the compendium).
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Solutions.

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1. Using the definition of L_p-norms we write

$$\begin{aligned}\|f\|_{L_1(a,b)} &= \int_a^b |f(x)|\,bx = \int_a^b 1\cdot|f(x)|\,bx \le \{\text{C-S}\} \le \Big(\int_a^b 1^2\,dx\Big)^{1/2}\Big(\int_a^b f^2(x)\,dx\Big)^{1/2}\\ &= \sqrt{b-a}\Big(\int_a^b f^2(x)\,dx\Big)^{1/2} = \sqrt{b-a}\|f\|_{L_2(a,b)}\\ &\le \sqrt{b-a}\Big(\int_a^b \max_{x\in[a,b]} f^2(x)\,dx\Big)^{1/2} = \sqrt{b-a}\Big(\int_a^b \max_{x\in[a,b]} |f(x)|^2\,dx\Big)^{1/2}\\ &= \sqrt{b-a}\max_{x\in[a,b]}|f(x)|\cdot\Big(\int_a^b dx\Big)^{1/2} = (b-a)\|f\|_{L_\infty(a,b)}.\end{aligned}$$

Thus, we have proved that

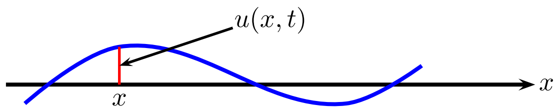
$$\|f\|_{L_1(a,b)} \le \sqrt{b-a}\|f\|_{L_2(a,b)} \le (b-a)\|f\|_{L_\infty(a,b)}.$$

If now $0<(b-a)\le 1$ then $0<\sqrt{b-a}\le 1$, then we get'

$$\|f\|_{L_1(a,b)} \le \|f\|_{L_2(a,b)} \le \|f\|_{L_\infty(a,b)}.$$

2. Some description: $\ddot u + b\dot u - u'' = 0$ models, e.g. the transversal oscillation of a wire fixed at its two endpoints, where $u(x,t)$ is the displacement coordinates.

The terms are corresponding to *inertia*, *friction (from the surrounding media)* and the resultant powers of all *tension and stress*, respectively.



Multiplying the equation by $\dot u$ and integrating in spatial variable over $[0,1]$ yields

$$\frac{1}{2}\frac{d}{dt}\Big(\|\dot u\|^2 + \|u'\|^2\Big) + b\|\dot u\|^2 = 0.$$

Hence considering the energy as

$$E(u) = \frac{1}{2}\|\dot u\|^2 + \frac{1}{2}\|u'\|^2,$$

we have that

$$\frac{d}{dt}E(u) = -b\|\dot u\|^2.$$

Thus

$$\frac{d}{dt}E(u) = 0, \quad \text{if } b = 0, \qquad \text{and} \quad \frac{d}{dt}E(u) \le 0 \quad \text{if } b > 0.$$

3. (a) Let $a = 1$ and use the following Galerkin orthogonality:

$$\int_0^1 (u - U)'v'\,dx = 0, \qquad \forall v \in V_h, \tag{1}$$

with v replaced by $2U$ (as in the second equality below) we get

$$\begin{aligned} ||(u-U)'||^2 &= \int_0^1 (u-U)'(u-U)'\,dx \\ &= \int_0^1 (u-U)'(u-U+2U)'\,dx = \int_0^1 (u-U)'(u+U)'\,dx \\ &= \int_0^1 (u')^2\,dx - \int_0^1 (U')^2\,dx = -2F(u) + 2F(U), \end{aligned} \tag{2}$$

where we have used the identities

$$2F(u) = ||u'||^2 - 2\int_0^1 fu\,dx = \{\text{with } w = u \text{ and } a = 1 \text{ in } (1)\} = -||u'||^2, \tag{3}$$

and similarly $2F(U) = -||U'||^2$.

(b) Recall that in the one dimensional case, we have the interpolation estimate,

$$||u' - U'|| \le C_i||hf||,$$

where C_i is an interpolation constant. This gives using (b) that

$$|F(U) - F(u)| \le C_i^2||hf||^2.$$

4. a) Multiplying the equation by u and performing partial integration we get

$$\int_\Omega \nabla u \cdot \nabla u + uu - \int_\Gamma n \cdot \nabla u u = \int_\Omega fu,$$

i.e.,

$$||\nabla u||^2 + ||u||^2 = \int_\Omega fu + \int_\Gamma gu \le ||f||||u|| + ||g||_\Gamma C_\Omega(||\nabla u|| + ||u||) \tag{4}$$

where $||\cdot|| = ||\cdot||_{L_2(\Omega)}$ and we have used the inequality $||u|| \le C_\Omega(||\nabla u|| + ||u||)$. Further using the inequality $ab \le a^2 + b^2/4$ we have

$$||\nabla u||^2 + ||u||^2 \le ||f||^2 + \frac{1}{4}||u||^2 + C||g||_\Gamma^2 + \frac{1}{4}||\nabla u||^2 + \frac{1}{4}||u||^2$$

which gives the desired inequality.

b) Consider the variational formulation

$$\int_\Omega \nabla u \cdot \nabla v + uv = \int_\Omega fv + \int_\Gamma gv, \tag{5}$$

set $U(x) = \sum U_j\psi_j(x)$ and $v = \psi_i$ in (7) to obtain

$$\sum_{j=1}^N U_j \int_\Omega \nabla\psi_j \cdot \nabla\psi_i + \psi_j\psi_i = \int_\Omega f\psi_i + \int_\Gamma g\psi_i, \quad i = 1, \dots, N.$$

This gives $AU = b$ where $U = (U_1, \dots, U_N)^T,\ b = (b_i)$ with the elements

$$b_i = h,\ i = 2, \dots, N-1, \quad b(N) = h/2, \quad b(1) = h/2 + 3,$$

and $A = (a_{ij})$ with the elements

$$a_{ij} = \begin{cases} -1/h + h/6, & \text{for } i = j+1 \quad \text{and } i = j-1 \\ 2/h + 2h/3, & \text{for } i = j \qquad\;\; \text{and } i = 2, \dots, N-1 \\ 0, & \text{else.} \end{cases}$$

5. Let V be the linear function space defined by

$$V_h := \{v : v \ \text{ is continuous in } \Omega, \ \ v = 0, \ \text{ on } \partial\Omega \setminus (\Gamma_1 \cup \Gamma_2)\}.$$

Multiplying the differential equation by $v \in V$ and integrating over Ω we get that

$$-(\Delta u, v) + (u, v) = (f, v), \qquad \forall v \in V.$$

Now using Green's formula we have that

$$\begin{aligned} -(\Delta u, \nabla v) &= (\nabla u, \nabla v) - \int_{\partial\Omega} (n \cdot \nabla u) v \, ds \\ &= (\nabla u, \nabla v) - \int_{\partial\Omega \setminus (\Gamma_1 \cup \Gamma_2)} (n \cdot \nabla u) v \, ds - \int_{\Gamma_1 \cup \Gamma_2} (n \cdot \nabla u) v \, ds \\ &= (\nabla u, \nabla v), \qquad \forall v \in V. \end{aligned}$$

Thus the variational formulation is:

$$(\nabla u, \nabla v) + (u, v) = (f, v), \qquad \forall v \in V.$$

Let V_h be the usual finite element space consisting of continuous piecewise linear functions satisfying the boundary condition $v = 0$ on $\partial\Omega \setminus (\Gamma_1 \cup \Gamma_2)$: The $cG(1)$ method is: Find $U \in V_h$ such that

$$(\nabla U, \nabla v) + (U, v) = (f, v) \qquad \forall v \in V_h$$

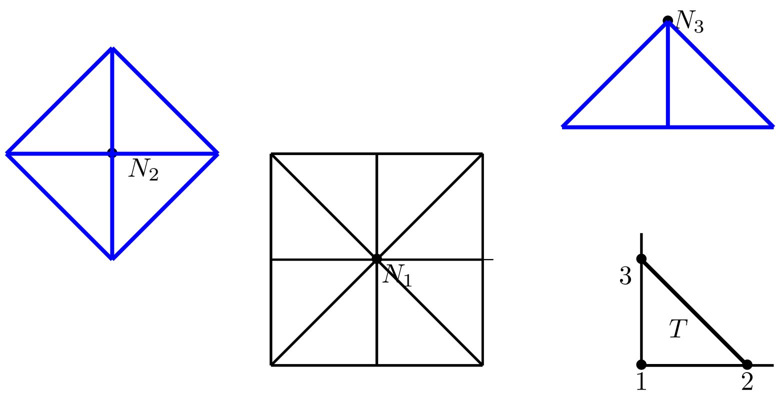
Making the "Ansatz" $U(x) = \sum_{j=1}^{3} \xi_j \varphi_j(x)$, where φ_i are the standard basis functions, we obtain the system of equations

$$\sum_{j=1}^{3} \xi_j \Big(\int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j \, dx + \int_\Omega \varphi_i \varphi_i \, dx \Big) = \int_\Omega f \varphi_j \, dx, \quad i = 1, 2, 3,$$

or, in matrix form,

$$(S + M)\xi = F,$$

where $S_{ij} = (\nabla\varphi_i, \nabla\varphi_j)$ is the stiffness matrix, $M_{ij} = (\varphi_i, \varphi_j)$ is the mass matrix, and $F_i = (f, \varphi_i)$ is the load vector.



We first compute the mass and stiffness matrix for the reference triangle T. The local basis functions are

$$\phi_1(x_1,x_2) = 1 - \frac{x_1}{h} - \frac{x_2}{h}, \qquad \nabla\phi_1(x_1,x_2) = -\frac{1}{h}\begin{bmatrix} 1 \\ 1 \end{bmatrix},$$
$$\phi_2(x_1,x_2) = \frac{x_1}{h}, \qquad \nabla\phi_2(x_1,x_2) = \frac{1}{h}\begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$\phi_3(x_1,x_2) = \frac{x_2}{h}, \qquad \nabla\phi_3(x_1,x_2) = \frac{1}{h}\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Hence, with $|T| = \int_T dz = h^2/2$,

$$m_{11} = (\phi_1,\phi_1) = \int_T \phi_1^2\,dx = h^2\int_0^1\int_0^{1-x_2}(1-x_1-x_2)^2\,dx_1dx_2 = \frac{h^2}{12},$$
$$s_{11} = (\nabla\phi_1,\nabla\phi_1) = \int_T |\nabla\phi_1|^2\,dx = \frac{2}{h^2}|T| = 1.$$

Alternatively, we can use the midpoint rule, which is exact for polynomials of degree 2 (precision 3):

$$m_{11} = (\phi_1,\phi_1) = \int_T \phi_1^2\,dx = \frac{|T|}{3}\sum_{j=1}^{3}\phi_1(\hat{x}_j)^2 = \frac{h^2}{6}\Big(0+\frac{1}{4}+\frac{1}{4}\Big) = \frac{h^2}{12},$$

where $\hat{x}_j$ are the midpoints of the edges. Similarly we can compute the other elements and obtain

$$m = \frac{h^2}{24}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \qquad s = \frac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

We can now assemble the global matrices M and S from the local ones m and s:

$$\begin{aligned}
M_{11} &= 8m_{22} = \frac{8}{12}h^2, & S_{11} &= 8s_{22} = 4,\\
M_{12} &= 2m_{12} = \frac{1}{12}h^2, & S_{12} &= 2s_{12} = -1,\\
M_{13} &= 0, & S_{13} &= 2s_{23} = 0,\\
M_{22} &= 4m_{11} = \frac{4}{12}h^2, & S_{22} &= 4s_{11} = 4,\\
M_{23} &= 2m_{12} = \frac{1}{12}h^2, & S_{23} &= 2s_{12} = -1,\\
M_{33} &= 2m_{22} = \frac{2}{12}h^2, & S_{33} &= 2s_{22} = 1.
\end{aligned}$$

The remaining matrix elements are obtained by symmetry $M_{ij} = M_{ji}$, $S_{ij} = S_{ji}$. Hence,

$$M = \frac{h^2}{12}\begin{bmatrix} 8 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \quad S = \varepsilon\begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} (1,\varphi_1) \\ (1,\varphi_2) \\ (1,\varphi_3) \end{bmatrix} = \begin{bmatrix} 8\cdot\frac{1}{3}\cdot\frac{1}{2} = \frac{4}{3} \\ 4\cdot\frac{1}{3}\cdot\frac{1}{2} = \frac{2}{3} \\ 2\cdot\frac{1}{3}\cdot\frac{1}{2} = \frac{1}{3} \end{bmatrix}.$$

6. See the book.

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