2021-03-03 **Exercise session w.7** Malin Nilsson

46) Consider the non-homog. heat equation in 1D with variable coefficient

$$\begin{cases} \dot u(x,t) - (a(x,t)u'(x,t))' = f(x,t) & x \in (0,1), t>0 \\ u(0,t) = u(1,t) = 0 & t>0 \\ u(x,0) = u_0(x) & x \in (0,1) \end{cases}$$

Formulate a $\underbrace{cG(1)}_{\text{in } x}\underbrace{dG(0)}_{\text{in } t}$ FEM for this prob.

Solution

Multiply DE by $v$, integrate over $(0,1)$, P.I. and integrate over an interval $I_n = (t_{n-1}, t_n)$

$\Rightarrow$ Variational formulation:

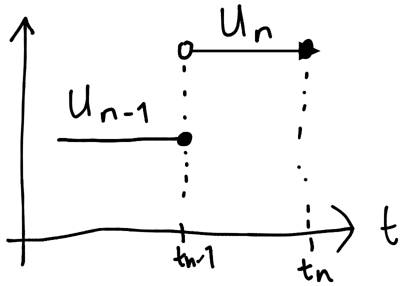
$$\int_{I_n}\int_0^1 (\dot u v + a u' v')\,dx\,dt = \int_{I_n}\int_0^1 f v\,dx\,dt$$

$$\forall \; v(x,t) : v(0,t) = v(1,t) = 0$$

cG(1) dG(0):

For each time interval $I_n$, length $= k_n$,

let $U(x,t) = U_n(x)$




$$U_n = \sum_{j=1}^{M} \xi_{n,j}\,\varphi_j(x)$$

$$U_n' = \sum_{j=1}^{M} \xi_{n,j}\,\varphi_j'(x)$$

$$\dot U = 0 + (U_n - U_{n-1})\,\delta_{t_{n-1}}$$

Test function in $t$: $1$

Test function in $x$: $\varphi_i$

$$VF \Rightarrow \int_{I_n}\int_0^1 aU'\varphi_i'\,dx\,dt + \int_{I_n}\int_0^1 (U_n - U_{n-1})\,\delta_{t_{n-1}}\varphi_i\,dx\,dt = \int_{I_n}\int_0^1 f\varphi_i\,dx\,dt$$

$$\left\{ \underline{\text{Def}}: \int_{I_n} \delta_{t_{n-1}}\,dt = 1 \right\}$$

$$\Rightarrow \int_{I_n}\int_0^1 aU'\varphi_i'\,dx\,dt + \int_0^1 (U_n - U_{n-1})\varphi_i\,dx = \int_{I_n}\int_0^1 f\varphi_i\,dx\,dt$$

$$\Rightarrow \sum_{j=1}^{M} \xi_{n,j} \underbrace{\int_{I_n}\int_0^1 a(x,t)\varphi_j'\varphi_i'\,dx\,dt}_{\substack{A \\ \text{(dep. on } t!)}} + \sum_{j=1}^{M}\left(\xi_{n,j} - \xi_{n-1,j}\right) \cdot \underbrace{\int_0^1 \varphi_j\varphi_i\,dx}_{M} = \underbrace{\int_{I_n}\int_0^1 f\varphi_i\,dx\,dt}_{b}$$

$$\Rightarrow \underline{(A+M)\,\xi_n = M\,\xi_{n-1} + b}$$

10.10) Consider the convection problem

$$\begin{cases} \dot{u} + \beta\cdot\nabla u + \alpha u = f & x\in\Omega,\ t>0 \\ u = g & x\in\Gamma_-,\ t>0 \\ u(x,0) = u_0(x) & x\in\Omega \end{cases}$$

with $\partial\Omega = \Gamma_+ \cup \Gamma_-$. Assume $\alpha - \frac{1}{2}\nabla\cdot\beta \geq c > 0$. Show the stability estimate:

$$\|u(\cdot,T)\|^2 + c\int_0^T \|u(\cdot,t)\|^2 dt + \int_0^T\int_{\Gamma_+} n\cdot\beta\, u^2\, ds\, dt \leq$$

$$\leq \|u_0\|^2 + \frac{1}{c}\int_0^T \|f(\cdot,t)\|^2 dt + \int_0^T\int_{\Gamma_-} |n\cdot\beta|\, g^2\, ds\, dt$$

$(\|\cdot\| = \|\cdot\|_{L^2(\Omega)})$

<u>Solution.</u> Multiply DE by $u$ and integrate over $\Omega$:

$$(*)\quad \int_\Omega \dot{u}u + \underbrace{\int_\Omega (\beta\cdot\nabla u)u} + \int_\Omega \alpha u^2 = \int_\Omega fu$$

$$\int_\Omega (\beta\cdot\nabla u)u = \frac{1}{2}\int_\Omega \beta\cdot\nabla(u^2) =$$

$$\left\{\text{Green's: } \int_\Omega \underset{\nabla\cdot\beta}{\underline{\Delta u}}\; \underset{u^2}{\underline{v}} = \int_{\partial\Omega} (\underset{\beta}{\underline{\nabla u}}\cdot n)\underset{u^2}{\underline{v}} - \int_\Omega \underset{\beta}{\underline{\nabla u}}\cdot\underset{\nabla(u^2)}{\underline{\nabla v}}\right\}$$

$$= \frac{1}{2}\int_{\partial\Omega}(\beta\cdot n)u^2 - \frac{1}{2}\int_\Omega u^2\nabla\cdot\beta$$

$\dot{u}u = \frac{1}{2}\frac{d}{dt}(u^2)$

$(*) \Rightarrow$

$$\frac{d}{dt}\int_\Omega u^2 + \int_{\Gamma_+} u^2(\beta\cdot n) + \int_{\Gamma_-} \underset{\uparrow\, g^2}{u^2}(\beta\cdot n) - \int_\Omega u^2\, \nabla\cdot\beta + 2\int_\Omega \alpha u^2 =$$

$$= 2\int_\Omega f u$$

$$\left\{ \int 2fu \le \left(\frac{f}{\sqrt{c}}\right)^2 + \left(\sqrt{c}\, u\right)^2 \right\}$$

$$\Rightarrow \frac{d}{dt}\|u\|^2 + \int_\Omega \underbrace{(2\alpha - \nabla\cdot\beta)}_{\ge 2c > 0} u^2 + \int_{\Gamma_+} u^2(\beta\cdot n) \le$$

$$\le -\int_{\Gamma_-} g^2(\beta\cdot n) + \frac{1}{c}\|f\|^2 + c\|u\|^2$$

$$\Rightarrow \frac{d}{dt}\|u\|^2 + c\|u\|^2 + \int_{\Gamma_+} u^2(\beta\cdot n)\, ds \le \frac{1}{c}\|f\|^2 + \int_{\Gamma_-} g^2|\beta\cdot n|$$

Integrate w.r.t $t$ from $0$ to $T$:

$$\int_0^T \frac{d}{dt}\|u\|^2 + c\int_0^T \|u(\cdot,t)\|^2 dt + \int_0^T\int_{\Gamma_+} u^2(n\cdot\beta)\, ds\, dt \le$$

$$\le \frac{1}{c}\int_0^T \|f(\cdot,t)\|^2 dt + \int_0^T\int_{\Gamma_-} g^2|n\cdot\beta|\, ds\, dt$$

$$\|u(\cdot,T)\|^2 + c\int_0^T \|u(\cdot,t)\|^2 dt + \int_0^T\int_{\Gamma_+} u^2(n\cdot\beta)\, ds\, dt \le$$

$$\le \|u_0\|^2 + \frac{1}{c}\int_0^T \|f(\cdot,t)\|^2 dt + \int_0^T\int_{\Gamma_-} g^2|n\cdot\beta|\, ds\, dt$$

10.4) Formulate the equation for cG(1)dG(1) for two-dim. heat eq. using the discrete Laplacian.

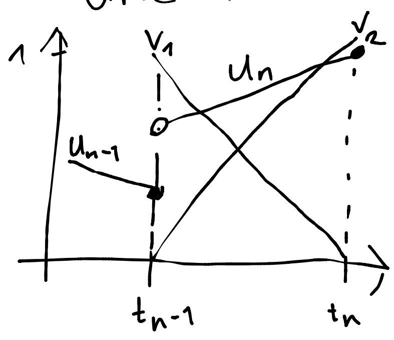
Solution 2-dim heat eq.:

$$\begin{cases} \dot{u}(x,t) - \Delta u(x,t) = f(x,t) & x \in \Omega,\ t>0 \\ u(x,t) = 0 & x \in \partial\Omega,\ t>0 \\ u(x,0) = u_0(x) & x \in \Omega \end{cases}$$

Let $I_n = (t_{n-1}, t_n]$. FEM: Find $u \in W_{k_n}^{(1)}$ s.t

$$\int_{I_n} \left( (\dot{u}, w) + (\nabla u, \nabla w) \right) dt = \int_{I_n} (f, w)\, dt$$

$\forall w \in W_{k_n}^{(1)} := \{ w :$ p.w. linear, cont. in space, p.w linear, discont. in $t \}$




Regard one space-time slab $S_n$

$$U_n(x,t) = \phi_n(x) \underbrace{\frac{t_n - t}{k_n}}_{= v_1} + \psi_n(x) \underbrace{\frac{t - t_{n-1}}{k_n}}_{v_2}$$

We'll use the discrete Laplacian and the projection formulation:

$(\nabla u, \nabla w) = -(\Delta_h u, w)$, $(P_h f, w) = (f, w)$ $\forall w$

$\Rightarrow$ FEM formulation:

$$\int_{I_n} \left( \dot{U} v_i - \Delta_h U v_i \right) dt = \int_{I_n} P_h f \, v_i \, dt \qquad i \in \{1,2\}$$

(using $v_1$ and $v_2$ as test fcns in $t$).

$$\dot{U}_n(x,t) = -\frac{1}{k_n} \phi_n(x) + \frac{1}{k_n} \psi_n(x) + \\ + \left( \phi_n(x) - P_h U_{n-1} \right) \delta_{t_{n-1}}$$

$\underline{i=1}$

$$\int_{I_n} \Big( \underbrace{\frac{1}{k_n}(\psi_n - \phi_n)}_{\text{indep. } t} v_1 + (\phi_n - P_h U_{n-1}) \underbrace{\delta_{t_{n-1}} v_1}_{\text{Def: } \int_{I_n} \delta_{t_{n-1}} v_1 = v_1(t_{n-1}) = 1} -$$

$\int_{I_n} v_1 = \frac{k_n}{2} = \int_{I_n} v_2$

$$\underbrace{- \Delta_h (\phi_n v_1 + \psi_n v_2) v_1}_{\int_{I_n} v_1 v_1 = \frac{k_n}{3} = \int_{I_n} v_2 v_2 \; , \; \int_{I_n} v_1 v_2 = \frac{k_n}{6}} \Big) dt = \int_{I_n} P_h f v_1 \, dt$$

$$\Rightarrow \frac{1}{k_n}(\psi_n - \phi_n) \frac{k_n}{2} + \phi_n - P_h U_{n-1} - \frac{k_n}{3} \Delta_h \phi_n - \frac{k_n}{6} \Delta_h \psi_n = \int_{I_n} P_h f \frac{t_n - t}{k_n} dt$$

$$\Rightarrow \frac{1}{2}(\psi_n - \phi_n) + \phi_n - P_h U_{n-1} - \frac{k_n}{6} \left( 2 \Delta_h \phi_n + \Delta_h \psi_n \right) = \int_{I_n} P_h f \frac{t_n - t}{k_n} dt$$

$\underline{i=2:}$

$$\int_{I_n} \Big( \frac{1}{k_n} (\psi_n - \phi_n) v_2 + (\phi_n - P_h U_{n-1}) \underbrace{\delta_{t_{n-1}} v_2}_{\int_{I_n} \delta_{t_{n-1}} v_2 = v_2(t_{n-1}) = 0} -$$

$$-\Delta_h(\phi_n v_1 + \psi_n v_2)v_2\Big)\,dt = \int_{I_n} P_h f\, v_2\, dt$$

$$\Rightarrow \frac{1}{k_n}(\psi_n - \phi_n)\cdot\frac{k_n}{2} - \frac{k_n}{6}\Delta_h \phi_n - \frac{k_n}{3}\Delta_h \psi_n = \int_{I_n} P_h f\, v_2\, dt$$

$$\Rightarrow \frac{1}{2}(\psi_n - \phi_n) - \frac{k_n}{6}(\Delta_h \phi_n + 2\Delta_h \psi_n) = \int_{I_n} P_h f\, \frac{t - t_{n-1}}{k_n}\, dt$$

So our cG(1)dG(1) formulation becomes:

$$\begin{cases} \left(\frac{1}{2}I - \frac{k_n}{6}\Delta_h\right)\psi_n + \left(\frac{1}{2}I - \frac{k_n}{3}\Delta_h\right)\phi_n = P_h U_{n-1} + \int_{I_n} P_h f\, \frac{t_n - t}{k_n}\, dt \\ \left(\frac{1}{2}I - \frac{k_n}{3}\Delta_h\right)\psi_n - \left(\frac{1}{2}I + \frac{k_n}{6}\Delta_h\right)\phi_n = \int_{I_n} P_h f\, \frac{t - t_{n-1}}{k_n}\, dt \end{cases}$$

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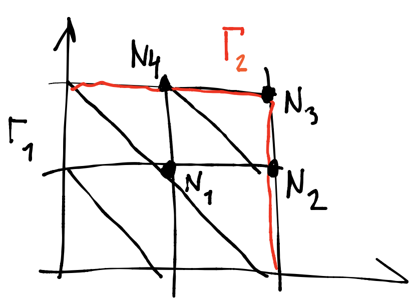
Exam 2017-03-15

4) In $\Omega = (0,2)^2$ with boundary $\Gamma = \partial\Omega$,

$$\begin{cases} -\Delta u + u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_1 = \Gamma \setminus \Gamma_2 \\ \left.\frac{\partial u}{\partial x_1}\right|_{x_1=2} = \left.\frac{\partial u}{\partial x_2}\right|_{x_2=2} = 1 & \text{on } \Gamma_2 = \{x_1 = 2\} \cup \{x_2 = 2\} \end{cases}$$

Determine S, M, b of cG(1) on:




$$m = \frac{h^2}{24}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad S = \frac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

## Solution.

VF: $(\nabla u, \nabla v) + (u, v) = (1, v) + (1, v)_{\Gamma_2} \quad \forall v \in V$

$$u_h = \sum_{j=1}^{4} \xi_j \varphi_j$$

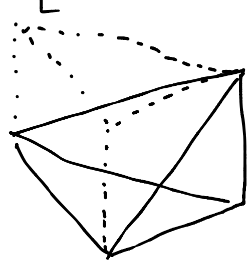
$$\sum_{j=1}^{4} \xi_j \Big( \underbrace{\int_\Omega \nabla\varphi_i \nabla\varphi_j \, dx}_{S} + \underbrace{\int_\Omega \varphi_i \varphi_j \, dx}_{M} \Big) = \int_\Omega \varphi_i \, dx + \int_{\Gamma_2} \varphi_i \, ds$$

$i \in \{1, 2, 3, 4\}$

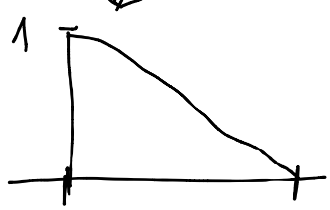
$$(S + M)\xi = b$$

## Load vector

$$b = \begin{bmatrix} (1,\varphi_1) + (1,\varphi_1)_{\Gamma_2} \\ (1,\varphi_2) + (1,\varphi_2)_{\Gamma_2} \\ (1,\varphi_3) + (1,\varphi_3)_{\Gamma_2} \\ (1,\varphi_4) + (1,\varphi_4)_{\Gamma_2} \end{bmatrix} = \begin{bmatrix} 6 \cdot \frac{1}{6} + 0 \\ 3 \cdot \frac{1}{6} + 2 \cdot \frac{1}{2} \\ 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{2} \\ 3 \cdot \frac{1}{6} + 2 \cdot \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 3/2 \\ 7/6 \\ 3/2 \end{bmatrix}$$



$$\text{Vol} = \frac{1 \cdot 1}{2} \cdot 1 \cdot \frac{1}{3} = \frac{1}{6}$$




$$\text{Area}: \frac{1 \cdot 1}{2} = \frac{1}{2}$$