Recall:

- Wave eq.

$$\begin{cases} u_{tt} - \Delta u = f & \text{in } \Omega \subset \mathbb{R}^d \\ u = 0 & \text{on } \partial\Omega \\ u(\cdot,0) = u_0 \ , \ u_t(\cdot,0) = v_0 & \text{in } \Omega \end{cases}$$

- (VF) Find u s.t. $(u_{tt}, v) + (\nabla u, \nabla v) = (f, v) \quad \forall v$

- (FE) Find u_h s.t. $(u_{h,tt}, \chi) + (\nabla u_h, \nabla \chi) = (f, \chi) \quad \forall \chi$

Th: $$\|u_h(t) - u(t)\|_{L^2(\Omega)} \leq C \cdot \left(|\pi_h u_0 - R_h u_0|_1 + \|\pi_h v_0 - R_h v_0\|_{L^2} \right) + C h^2 \left(\|u(t)\|_{H^2} + \int_0^t \|u_{tt}(s)\|_{H^2} \, ds \right).$$

Proof:

- $$u_h - u = \underbrace{(u_h - R_h u)}_{\theta} + \underbrace{(R_h u - u)}_{\rho}$$

- $$\|\rho(t)\|_{L^2} + h|\rho(t)|_1 \leq C \cdot h^2 \|u(t)\|_{H^2}$$

$$\|\rho_t(t)\|_{L^2} \leq C \cdot h^2 \|u_t(t)\|_{H^2}, \quad \|\rho_{tt}(t)\|_{L^2} \leq C \cdot h^2 \|u_{tt}(t)\|_{H^2}.$$

- $$\|\theta_t(t)\|_{L^2}^2 + \underbrace{|\theta(t)|_1^2}_{\geq 0} \leq \|\theta_t(0)\|_{L^2}^2 + |\theta(0)|_1^2 + 2\int_0^t \|\rho_{tt}(s)\|_{L^2} \underbrace{\|\theta_t(s)\|_{L^2}}_{\leq \max_{0\leq s\leq t} \|\theta_t(s)\|_{L^2}} ds$$

$t \leq T$

$2xy \leq 2x^2 + \frac{1}{2}y^2$

$$\leq \|\theta_t(0)\|_{L^2}^2 + |\theta(0)|_1^2 + 2\left(\int_0^T \|\rho_{tt}(s)\|_{L^2} ds\right)^2 +$$

$$+ \underbrace{\frac{1}{2}\left(\max_{0\leq s\leq t} \|\theta_t(s)\|_{L^2}\right)^2}_{\text{to left}} \qquad \forall t \in [0,T]$$

This gives:

$$\frac{1}{2}\left(\max_{0\leq s\leq t} \|\theta_t(s)\|_{L^2}\right)^2 \leq \|\theta_t(0)\|_{L^2}^2 + |\theta(0)|_1^2 +$$

$$+ 2\left(\int_0^T \|\rho_{tt}(s)\|_{L^2} ds\right)^2$$

All together, we obtain:

$$\underbrace{\|\theta_t(t)\|_{L^2}^2}_{\geq 0} + \underbrace{|\theta(t)|_1^2}_{\|\nabla\theta(t)\|_{L^2}^2} \leq 2\|\theta_t(0)\|_{L^2}^2 + 2|\theta(0)|_1^2 +$$

$$+ 4\left(\int_0^T \|\rho_{tt}(s)\|_{L^2}\, ds\right)^2$$

This is valid for all $t \in [0,T]$, in particular $t = T$, arbitrary.

- Finally, we can bound the error of the FEM as follows:

$$\|u_h(t) - u(t)\|_{L^2} \overset{\Delta}{\leq} \|\theta(t)\|_{L^2} + \|\rho(t)\|_{L^2} \leq$$

Poincaré ineq.

$$\leq C \cdot |\theta(t)|_1 + \|\rho(t)\|_{L^2} \leq$$

$$\leq C \cdot \left(\|\theta_t(0)\|_{L^2} + |\theta(0)|_1 + \int_0^t \|\rho_{tt}(s)\|_{L^2}\, ds \right) + \|\rho(t)\|_{L^2} \leq$$

above second point

$$\leq C \cdot \left(\|\Pi_h v_0 - R_h v_0\|_{L^2} + |\underbrace{u_h(0)}_{\Pi_h u_0} - R_h u_0|_1 + \int_0^t C h^2 \|u_{tt}(s)\|_{H^2}\, ds \right) +$$

Def $\theta(0)$, $\theta_t(0)$ above first point

$$+ C h^2 \|u(t)\|_{H^2} \leq$$

$$\leq C \cdot \left(\|\Pi_h v_0 - R_h v_0\|_{L^2} + |\Pi_h u_0 - R_h u_0|_1 + h^2 \int_0^t \|u_{tt}(s)\|_{H^2}\, ds + h^2 \|u(t)\|_{H^2} \right).$$

∎

Chapter XIV: The finite element

Goal: Give formal definition of FEM

1) Formal definition of finite element:

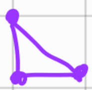
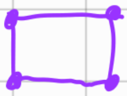
Def: (Ciarlet, 1978) A finite element consits of a triplet (K, P, Σ_1), where

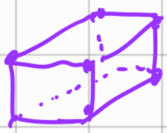
- $K \subset \mathbb{R}^d$ is polygon
- P is a polynomial function space (of finite dimension)
- Σ_1 is a (unisolvent) set of linear functionals on P

$\Sigma_1 = \{L_1, L_2, \dots, L_n\}$, where $n = \dim(P)$.

Ex: • $K \rightsquigarrow$ element domain

1d  line, ...

2d  triangle,  quadrilateral, ...

3d  bricks, ...

...

- $P \rightsquigarrow$ space of shape functions

  $P^{(1)}(K) \rightsquigarrow$ set of polynomials of degree ≤ 1 on K

  $P^{(2)}(K) \rightsquigarrow$ polym. degree ≤ 2 on K

  ...

- $\Sigma' \rightsquigarrow$ set of nodal variables;

  This uniquely specify the shape functions on each element K as well as the behaviour of these functions between adjcent elements.

<u>Ex</u>: 1d Lagrange $P^{(\ell)}$ elements:

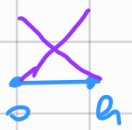
Let $a < b$, consider distinct points

$$a = x_0 < x_1 < x_2 < \dots < x_\ell = b.$$

hat functions

Let $K = [a,b]$

[early



]

$\ell = 1$

Let $P = P^{(\ell)}(K) \rightsquigarrow$ set of polynomials of degree $\leq \ell$ defined on $K = [a,b]$.

Let $\Sigma' = \{L_0, L_1, L_2, \dots, L_\ell\}$, where $L_j : P \to \mathbb{R}$ are defined by $L_j(f) = f(x_j) \quad \forall f \in \mathbb{P}$.

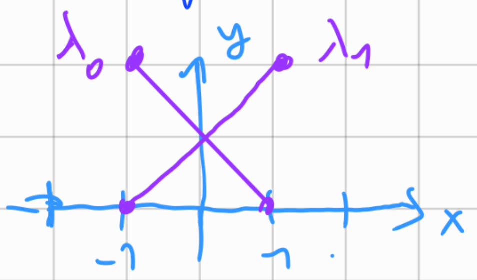
Now, we know that Lagrange polynomials

$$\lambda_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{\ell} \frac{x-x_j}{x_i-x_j}$$

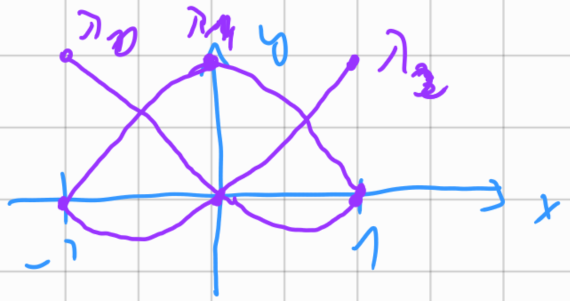
satisfy $\lambda_i(x_j) = \delta_{ij} = \begin{cases} 0 & \text{if } i\neq j \\ 1 & \text{if } i=j \end{cases}$

Observe then, that $L_j(\lambda_i) = \lambda_i(x_j) = \delta_{ij}$

$\rightsquigarrow \dots \longrightarrow$ this tells us that λ_j can be taken as a basis for $\mathbb{P}$ and this exactly the hat functions in the case $\ell = 1$:



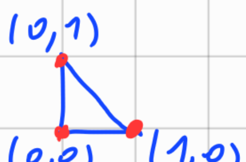
$\ell = 1$
2 shape functions



$\ell = 2$
3 shape functions

<u>Ex</u>: 2d linear Lagrange elements:

Consider the reference triangle K



The space $P = P^{(1)}(K)$ ⇝ polyn. of degree ≤ 1 on K.

$\Sigma'_1 = \{ L_1, L_2, L_3 \}$, where $L_j : P \to \mathbb{R}$ are defined by

$$L_1(f) = f(0,0), \quad L_2(f) = f(1,0), \quad L_3(f) = f(0,1)$$

Next, we determine the shape functions as follows:

$$S_j(x,y) = a_j + b_j x + c_j y \quad \text{for } j = 1,2,3, \ (x,y) \in K.$$

and

$$L_i(S_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{else} \end{cases} \qquad \text{for } i,j = 1,2,3$$

The above provides a linear system of equations

with solutions

$$S'_1(x,y) = 1 - x - y$$

$$S'_2(x,y) = x$$

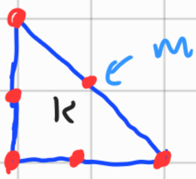
$$S'_3(x,y) = y$$

Remember hat functions / shape functions from previous chapters

For 2d elements, one can also consider

the following FE:

Ex:



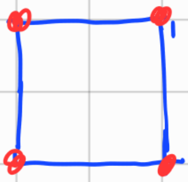
$\rightsquigarrow$ quadratic Lagrange elements

$$S_1(x,y) = 1 - 3x - 3y + 2x^2 + 4xy + 2y^2$$

$$S_2, S_3, S_4, S_5,$$

$$S_6(x,y) = 4x - 4x^2 - 4xy$$

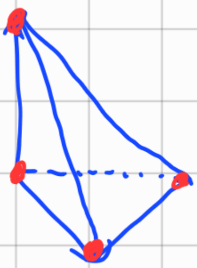
Ex:



$\rightsquigarrow$ bilinear quadrilateral elements

↳ $(a+bx)(c+dx)$

Ex: In 3d:

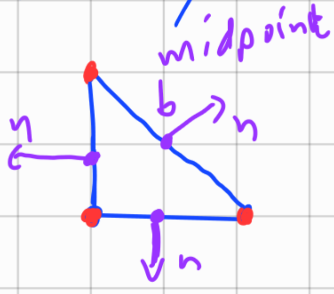


$\rightsquigarrow$ linear Lagrange element on tetrahedron

Rem: • One can have other polynomials than Lagrange, for instance Hermite polynomials ($\rightsquigarrow C^{(1)}(\Omega)$ element instead of $C^{(0)}(\Omega)$ element with Lagrange)

- One can get more exotic FE:

Morley elements / Morley–Wang–Xu / MWX



(m=d=2)

$L_i(f) = f(\cdot)$ $\quad i = 1,2,3$

midpoints

$L_{i+3}(f) = n \cdot \nabla f(\cdot)$ $\quad i = 1,2,3$

$(0,-1)\cdot \nabla f(1/2,0))$ $\qquad f = 1+x+y$

⇝ get approximation in H^m in $\mathbb{R}^d$

⇝ solid mechanics

2) Variational crimes:

In theory, we consider

(VF) Find $u \in U$ s.t. $a(u,v) = \ell(v)$ $\forall v \in V$

In reality, one works with the problem

(FE) Find $u_h \in U_h$ s.t. $a_h(u_h,\chi) = \ell_h(\chi)$ $\forall \chi \in V_h$,

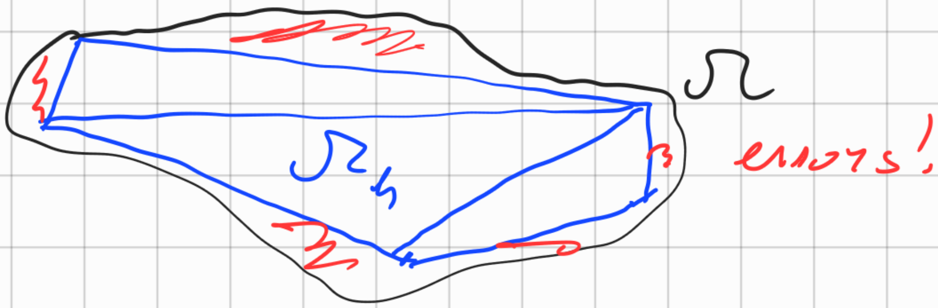
where the index h denotes, f.ex.,

→ errors from approximations of integrals

$$\int_K f(x)\,dx \approx \frac{f(N_i)}{3}|K|$$

errors!!

→ error in the mesh of Ω:



↳ One has to deal with these variational crimes (Strang, 1972) ...