Examination, 15 March 2021
MVE455

Read this before you start!
Aid: Anything but collaboration.
If something is not clear you can ask to talk to me over zoom.
Read all questions first and start to answer the ones you like most.
Answers may be given in English, French, German or Swedish.
Write down all the details of your computations clearly so that each steps are easy to follow.
Do not randomly display equations and hope for someone to find the correct one. Justify your answers!
Write clearly what your solutions are and in the nicest possible form.
Don't forget that you can verify your solution in some cases.
Write your cid or first numbers of your personnummer.
Use a proper pen, order your answers, use an app like camscanner or equivalent, and check your final scan before uploading it.
The test has 3 pages and a total of 20 points.
Valid bonus points will be added to the total score if needed.
You will be informed when the exams are corrected.

"I assure that I did this exam on my own without getting help from any other person and that I formulated all the solutions myself."
Check the box □
Good luck!

Some exercises were taken from, or inspired by, materials from *P.E. Farrell*, *K. Larsson*, *P.J. Olver*.

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1. Short questions with short motivated answers:

   (a) Is the partial differential equation (defined in $2d$)

   $$u_{xx}(x,y) + u_{yy}(x,y) = x^2 + y^3$$

   parabolic? (1p)

   (b) Which type of error estimates can be used for adaptivity? (1p)

   (c) Give one property of the hat functions that is reflected in the final matrices coming from a Galerkin discretisation of a BVP or PDE. (1p)

2. Let $i = \sqrt{-1}$ and a given initial value u_0. Consider the linear Schrödinger equation (for $x \in [0,1]$ and $t \in [0,T]$)

$$\begin{aligned} iu_t(x,t) - u_{xx}(x,t) &= 0 \\ u(x,0) &= u_0(x) \end{aligned}$$

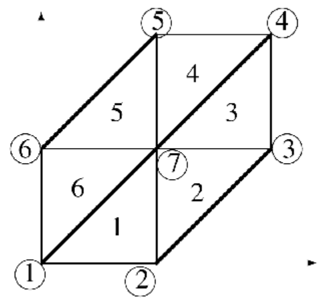

Figure 1: Courtesy from N. Kopteva.

with homogeneous Dirichlet boundary conditions.

Show that the L^2-norm of the solution $||u(\cdot,t)||_{L^2(0,1)}$ is a conserved quantity for all time $t>0$. This can be proven directly or you may start by writing the complex-valued function u as $u = v + iw$ with two real-valued functions v and w and get a system of linear PDEs for (v,w). (2p)

Hint: For complex-valued functions f,g the L^2-inner product reads

$$(f,g) = \int f(x)\bar{g}(x)\,dx.$$

Try to inspire yourself with what we did for the linear wave equation in the lecture.

3. Let $\Omega \subset \mathbb{R}^2$ be a nice domain and b,c,f nice (non-zero) scalar functions defined on Ω. Consider the problem

$$-\nabla\cdot(c(x)\nabla u(x)) + b(x)u(x) = f(x) \quad \text{in} \quad \Omega$$
$$\frac{\partial u}{\partial n}(x) = 0 \quad \text{on} \quad \partial\Omega,$$

where we have used the notations $x = (x_1,x_2)\in\mathbb{R}^2$, $\frac{\partial u}{\partial n}(x)$ for the normal derivative $n\cdot\nabla u(x)$, and n for the unit outward normal vector.

(a) Write down the variational formulation of this problem. If you cannot apply Green's formula directly, you may adapt formula 9.2.9 from the book. (1p)

(b) Give conditions on b and c that guarantee existence and uniqueness of the solution to the variational problem (we suppose that f is nice enough). (3p)

(c) Consider a triangulation of Ω with mesh size h. Formulate the piecewise linear finite element problem for the above PDE. (1p)

(d) Consider the uniform triangulation of a domain Ω consisting of reference triangles (with nodes $(0,0)$, $(1,0)$, and $(0,1)$) from Figure 1. Consider the above problem with $b(x) = f(x) = 1$. Compute the entry of the matrix coming from the term $b(x)u(x)$ in the above PDE for the node ② as well as the entry of the load vector for the node ①. (4p)

4. Let $\Omega = (0,1)$ and $f \in L^2(\Omega)$. Consider the problem

   Find $u \in H_0^1(\Omega)$ such that $\int_\Omega u'(x)v'(x)\,\mathrm{d}x = \int_\Omega f(x)v(x)\,\mathrm{d}x$ for all $v \in H_0^1(\Omega)$.

   Let $u_h \in V_h^0$ be the corresponding cG(1) approximation to u on a uniform partition with mesh size h. Consider then the auxiliary problem

   Find $\zeta \in H_0^1(\Omega)$ such that $\int_\Omega \zeta'(x)v'(x)\,\mathrm{d}x = \int_\Omega (u(x) - u_h(x))\, v(x)\,\mathrm{d}x$ for all $v \in H_0^1(\Omega)$.

   (a) Using the above auxiliary problem and Galerkin's orthogonality, first show that

   $$\|u - u_h\|_{L^2(\Omega)}^2 = \int_\Omega (u(x) - u_h(x))'(\zeta(x) - \pi_h\zeta(x))'\,\mathrm{d}x,$$

   where we recall that $\pi_h\zeta \in V_h^0$ denotes the continuous piecewise linear interpolant of ζ. (2p)

   (b) Next, using an interpolation error estimate (observe that $\zeta \in H^2(\Omega)$ since $-\zeta'' = u - u_h$), show the following error estimate

   $$\|u - u_h\|_{L^2(\Omega)} \leq Ch\,\|(u - u_h)'\|_{L^2(\Omega)}\,.$$

   (2p)

   (c) Finally, from an a priori error estimate from the lecture, obtain the improved error estimate for Poisson's equation

   $$\|u - u_h\|_{L^2(\Omega)} \leq Ch^2\,\|f\|_{L^2(\Omega)}\,.$$

   (2p)