Examination, 9 June 2021
TMA372 and MMG800

Read this before you start!
Aid: Anything but collaboration.
If something is not clear you can ask to talk to me over zoom.
In case of problems, I'll try to contact all of you via canvas.
Read all questions first and start to answer the ones you like most. Some parts of an exercise may be independent of the others.
Answers may be given in English, French, German or Swedish.
Write down all the details of your computations clearly so that each steps are easy to follow.
Do not randomly display equations and hope for someone to find the correct one. Justify your answers!
Write clearly what your solutions are and in the nicest possible form.
Don't forget that you can verify your solution in some cases.
Write your cid or first numbers of your personnummer.
Use a proper pen, order your answers, use an app like camscanner or equivalent, and check your final scan before uploading it. Check also the uploaded version.
The test has 4 pages and a total of 30 points.
Valid bonus points will be added to the total score if needed.
You will be informed when the exams are corrected.

"I assure that I did this exam on my own without getting help from any other person and that I formulated all the solutions myself."
Check the box □
Good luck!

Some exercises were taken from, or inspired by, materials from *P.E. Farrell*, *M.J. Gander*, *F. Kwok*, *K. Larsson*, *P.J. Olver*, *R. Rannacher*.

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1. Let $p \in C^1([0,1])$ with $p > 0$ and $r, f \in C^0([0,1])$ with $r > 0$ and $A, B \in \mathbb{R}$. Consider the problem (in strong form): Find $u \in C^2(0,1) \cap C^1([0,1])$ such that

$$\begin{cases} -(p(x)u'(x))' + r(x)u(x) = f(x), \quad \text{in} \quad (0,1), \\ -p(0)u'(0) = A, \\ p(1)u'(1) = B. \end{cases}$$

   Find the corresponding minimization problem. (2p)

   <u>*Hint*</u>*: You may first derive the variational formulation of this problem.*

2. Consider the ordinary differential equation

$$\begin{cases} \dot{y}(t) = f(y(t)) \\ y(0) = y_0. \end{cases}$$

Write the above as an integral equation (justify). Let $k > 0$ and consider this integral equation on the small interval $[0, k]$. Apply the trapezoidal rule to the integral. From this last computation, using $y_1 \approx y(k)$, derive the Crank–Nicolson scheme for solving the above ODE. (3p)

3. Let $\Omega = (0, 1)$ and consider a nice function $f\colon \Omega \to \mathbb{R}$. We want to solve the advection-diffusion equation

$$-\mu u_{xx} + 3u_x = f, \quad \text{in } \Omega = (0, 1),$$

with $\mu > 0$ and homogeneous Dirichlet boundary conditions.

   (a) Derive the variational form of the problem. (2p)

   (b) Use hat functions to define a Galerkin approximation (cG(1)) of this problem and compute the tridiagonal matrix in the final linear system of equation $A\zeta = b$. (3p)

4. An alternative to triangular elements is to use piecewise bi-affine functions ($w(x, y) = \alpha + \beta x + \gamma y + \delta xy$) on the unit square for instance. Denote by P_j, $j = 1, \ldots, 4$, the vertices of this unit square. Show that for $\ell = 1$ (this is in fact true for each $\ell = 1, \ldots, 4$ but the argument is the same), there exists a unique bi-affine function $w_\ell(x, y)$ defined on the unit square that has the values $w_\ell(P_j) = \delta_{\ell,j}$. Write down the explicit form of the bi-affine function $w_1(x, y)$. (3p)

5. Let V be an Hilbert space with inner product and norm denoted by $(\cdot, \cdot)_V$ and $\|\cdot\|_V$. On this space, let a bilinear form $a(\cdot, \cdot)$ and a functional $\ell(\cdot)$ verifying the assumptions of Lax–Milgram that we recall: There exist $\alpha > 0, \beta \geq 0, \kappa > 0$ such that

$$\begin{aligned} |a(u, v)| &\leq \alpha \|u\|_V \|v\|_V \quad \forall u, v \in V \\ a(u, u) &\geq \kappa \|u\|_V^2 \quad \forall u \in V \\ |\ell(v)| &\leq \beta \|v\|_V \quad \forall v \in V. \end{aligned}$$

Consider the variational problem: Find $u \in V$ such that

$$a(u, \varphi) = \ell(\varphi) \quad \forall \varphi \in V.$$

Let now $V_h \subset V$ be a finite dimensional subspace of V and $u_h \in V_h$ the solution to Galerkin's equation

$$a(u_h, \varphi_h) = \ell(\varphi_h) \quad \forall \varphi_h \in V_h.$$

   (a) Show that the discrete solution u_h exists and is unique in V_h. (2p)

   (b) Show the following bound

$$\|u - u_h\|_V \leq \frac{\alpha}{\kappa} \left\|u - \varphi_h\right\|_V$$

   for all $\varphi_h \in V_h$. (3p)

   Hint: Look and use the assumptions of Lax–Milgram.

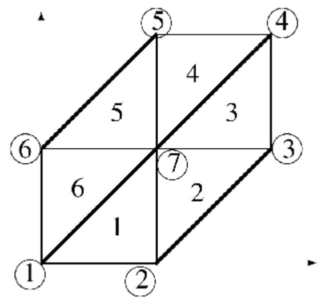

Figure 1: Courtesy from N. Kopteva.

(c) Consider a regular domain $\Omega \subset \mathbb{R}^2$ and $f \in L^2(\Omega)$. Apply the above results to show convergence of Galerkin's approximation of the diffusion-convection problem

$$\begin{aligned} -\Delta u + \partial_x u = f \quad &\text{in} \quad \Omega, \\ u = 0 \quad &\text{on} \quad \partial\Omega, \end{aligned}$$

where one denotes $u = u(x, y)$. Feel free to assume that one can apply Lax–Milgram in the present situation so that one can use the results from the above items. (2p)

(d) Consider the above problem with $f = 1$ on a uniform triangulation of the domain Ω consisting of reference triangles (with nodes $(0, 0)$, $(1, 0)$, and $(0, 1)$) from Figure 1. Compute the entry of the matrix coming from the term $\partial_x u$ in the above PDE for the node ② as well as the entry of the load vector for the node ①. (4p)

6. Let $\Omega = (0, 1)$, $f \in L^2(\Omega)$, and the space $H^2_0(\Omega) = \{v \in H^2(\Omega) : v(0) = v(1) = 0, v'(0) = v'(1) = 0\}$. Consider the variational problem

$$\text{Find } u \in H^2_0(\Omega) \text{ such that } \int_\Omega u''(x)v''(x)\,\mathrm{d}x = \int_\Omega f(x)v(x)\,\mathrm{d}x \quad \forall v \in H^2_0(\Omega).$$

(a) First, show that

$$\|v'\|_{L^2(\Omega)} \leq C\,\|v''\|_{L^2(\Omega)}$$

for all $v \in H^2_0(\Omega)$. (2p)

Hint: One may use Poincaré inequality.

(b) Observing that $H^2_0(\Omega) \subset H^1_0(\Omega)$ and the above inequality, show that

$$\|v\|_{H^2(\Omega)} \leq C\,\|v''\|_{L^2(\Omega)}$$

for all $v \in H^2_0(\Omega)$. (2p)

Hint: One may use Poincaré inequality.

(c) Let a uniform partition of Ω with mesh size h. Consider now the finite element space $W_{0h} = \{v_h \in C^1(\bar{\Omega}) : \ v_h|_{[x_j,x_{j+1}]} \in P^3 \quad \text{and} \quad v_h(0) = v_h(1) = 0, v_h'(0) = v_h'(1) = 0\}$, where P^3 is the set of polynomials of degree ≤ 3 on an element. Denote by u_h the corresponding finite element approximation to the above given variational problem.
Using the fact that

$$\|u - u_h\|_{H^2(\Omega)} \leq C \|u - \pi_h u\|_{H^2(\Omega)} \quad \text{and} \quad \|(v - \pi_h v)''\|_{L^2(\Omega)} \leq Ch^2 \|v''''\|_{L^2(\Omega)}$$

for $v \in H^4(\Omega) \cap H_0^2(\Omega)$ and the interpolation operator π_h and the fact that $u'''' = f$ (in $L^2(\Omega)$), show the final error estimate

$$\|u - u_h\|_{H^2(\Omega)} \leq Ch^2 \left\|f\right\|_{L^2(\Omega)}. \qquad \text{(2p)}$$