Exam, 9 June 2021

TMA372, MMG800, MVE455

(1) VF: Find $u \in H^1(0,1)$ s.t. $\int_0^1 p u' v' \, dx + \int_0^1 r u v \, dx = \int_0^1 f v \, dx + A v(0) + B v(1)$ $\forall v \in H^1(0,1)$ (1)

MP: Define $F(v) := \frac{1}{2}(pv', v')_{L^2} + \frac{1}{2}(rv, v) - (f, v) - A v(0) - B v(1)$

$\hookrightarrow$ Find $u \in H^1(0,1)$ s.t. $F(u) \to$ minimal (1)

(2) IVP on $[0,k]$ reads $y(k) \underset{\uparrow}{=} y_0 + \int_0^k f(y(t))\,dt \underset{\uparrow}{\approx} y_0 + \frac{k}{2}\left(f(y_0) + f(y(k))\right)$

Fund. th. calculus (1) Trapezoidal rule (1)

Use $y_1 \approx y(k)$ to get $y(k) \approx y_1 = y_0 + \frac{k}{2}\left(f(y_0) + f(y_1)\right)$ i.e. CN! (1)

(3)
a) Hom. Dirichlet BC $\rightsquigarrow$ use $H_0^1(0,1)$ (1)

VF: Find $u \in H_0^1$ s.t. $\mu \int_0^1 u_x v_x \, dx + 3 \int_0^1 u_x v \, dx = \int_0^1 f v \, dx$ $\forall v \in H_0^1$ (1)

b) Matrix has entries

$A_{ij} = \mu(\varphi_i', \varphi_j')_{L^2} + 3(\varphi_i', \varphi_j)_{L^2} = \text{tridiag}\left(-1 - \frac{\gamma}{2}, 2, -1 + \frac{\gamma}{2}\right) \cdot \frac{\mu}{h}$, where $\gamma := \frac{3h}{\mu}$ (1) (1) (1)

Use def φ_j, φ_j' $(j = 1, \dots, n)$ and compute the integrals

(see also lecture)

(4) To find $w_1(x,y)$ $\rightsquigarrow$ solve a linear system:

Node $(0,0)$: $1 \doteq \alpha$

Node $(1,0)$: $0 = \alpha + \beta$

Node $(0,1)$: $0 = \alpha + \gamma$

Node $(1,1)$: $0 = \alpha + \beta + \gamma + \delta$

δ_{ij} $W_1(\text{Nodes})$ (1)

$\rightarrow \alpha = 1, \beta = -1 = \gamma, \delta = 1$ (1)

$\Rightarrow w_1(x,y) = 1 - x - y + xy$ (1)

(5) a) $V_h \subset V$ Hilbert $\rightarrow$ one can apply LM to show $\exists!$ of u_h! (2)

b) For $u \in V$ and $u_h \in V_h$, one has $a(u, \varphi_h) = \ell(\varphi_h)$ and $a(u_h, \varphi_h) = \ell(\varphi_h)$

$\forall \varphi_h \in V_h \subset V \Rightarrow a(u - u_h, \varphi_h) = 0$ $\forall \varphi_h \in V_h \subset V$

Then, $\kappa' \|u-u_h\|_V^2 \underset{\text{ass. LM}}{\leq} a(u-u_h, u-u_h) = a(u-u_h, u) + \underbrace{a(u-u_h, \varphi_h)}_{0 \text{ above}} - a(u-u_h, \varphi_h)$ (1)

$= a(u-u_h, u-\varphi_h) \underset{\text{ass. LM}}{\leq} \alpha \|u-u_h\|_V \|u-\varphi_h\|_V$ (1)

$\leadsto \kappa' \|u-u_h\|_V \leq \alpha \|u-u_h\|_V$ and $\|u-u_h\|_V \leq \frac{\alpha}{\kappa'} \|u-\varphi_h\|_V \quad \forall \varphi_h \in V_h$ (1)

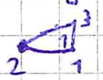
c) Part b) gives, for $V = H_0^1(\Omega)$,

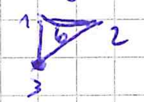
$\|u-u_h\|_V \leq \frac{\alpha}{\kappa'} \|u-\varphi_h\|_V \leq \frac{\alpha}{\kappa'} \|u - \underbrace{I_h u}_{\text{interpolant}}\|_V \longrightarrow 0$ as $h \to 0$ (1)+(1)

d) Hat fct $\hat{\varphi}_1(x,y) = 1-x-y$, $\hat{\varphi}_2(x,y) = x$, $\hat{\varphi}_3(x,y) = y$

Load vector at node ①:

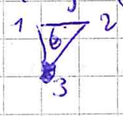
$\int_\Omega 1 \cdot \varphi_1 = \int 1 \cdot \hat{\varphi}_2 + \int 1 \cdot \hat{\varphi}_3 = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$ // (1)+(1)





Matrix at node ②:

$\int_\Omega \partial_x \varphi_2(x,y)\, \varphi_2(x,y) = \int (\partial_x \hat{\varphi}_1) \hat{\varphi}_1 + \int (\partial_x \hat{\varphi}_3) \hat{\varphi}_3 = -\frac{1}{6} + 0 = -\frac{1}{6}$ (1)+(1)



⑥

a) For $v \in H_0^2(\Omega)$, one has $v' \in H_0^1(\Omega)$ by def of Sobolev spaces. (1)

Apply Poincaré to v' and get $\|v'\|_{L^2} \leq C \cdot \|v''\|_{L^2}$ (1)

b) Since $H_0^2(\Omega) \subset H_0^1(\Omega)$, one can apply Poincaré to $v \in H_0^2(\Omega)$ (1)

and get $\|v\|_{L^2} \leq C \cdot \|v'\|_{L^2} \underset{\text{above}}{\leq} C \cdot \|v''\|_{L^2}$ (1)

Now, def H^2, $\|v\|_{H^2} \leq C \cdot \|v''\|_{L^2}$

c) Conclusion.

$\|u-u_h\|_{H^2(\Omega)} \overset{\text{Hint}}{\leq} C \cdot \|u - \Pi_h u\|_{H^2(\Omega)} \overset{\text{by def + b)}}{\leq} C \cdot \|(u-\Pi_h u)''\|_{L^2} \leq C \cdot h^2 \|u''''\|_{L^2}$ (1)

$\underset{u'''' = f \in L^2 \text{ as } u \in H^4 \cap H_0^1}{\leq} C \cdot h^2 \|f\|_{L^2}$ (1)