Exam, 26 August 2021 (A)

TMA 372, MMG 800, MVE455

(1) See summary: compute discriminant d

a) $B=1, D=-1 \rightsquigarrow d=1^2=1>0 \rightsquigarrow$ hyperbolic (1)

b) $A=1, B=1, C=y \Rightarrow d=1^2-4\cdot 1\cdot y=1-4y \Rightarrow \begin{cases} y>1/4 & \text{ell.} \\ y=1/4 & \text{parab.} \\ y<1/4 & \text{hyp.} \end{cases}$ (1)

c) $A=2, B=4, C=2 \rightsquigarrow d=16-4\cdot 2\cdot 2=0 \rightsquigarrow$ parab. (1)

(2) Hint $\Rightarrow f(x)=\underbrace{f(x_i)}_{0}+\int_{x_i}^{x} 1\cdot f'(z)\,dz$ for $x\in[x_i,x_{i+1}]$ and $i=0,1,\ldots,N$

Next, $|f(x)|=\left|\int_{x_i}^{x} 1\cdot f'(z)\,dz\right| \overset{C-S}{\le} \sqrt{\int_{x_i}^{x} 1^2\,dz}\cdot\sqrt{\int_{x_i}^{x}|f'(z)|^2\,dz} \le \sqrt{h_{max}}\cdot\sqrt{\int_{x_i}^{x}|f'(z)|^2\,dz}$ (1)

Finally, $\|f\|_{L^2}^2=\sum_{i=0}^{N-1}\int_{x_i}^{x_{i+1}}|f(x)|^2\,dx \le h_{max}\sum_{i=0}^{N-1}\int_{x_i}^{x_{i+1}}|f'(z)|\,dz = h_{max}^2\|f'\|_{L^2}^2$ (Def h_{max}) (1)

(3) $\int_0^1 u(x)\,dx \underset{\text{Trapezoidal rule}}{\approx} \frac{1-0}{2}(u(0)+u(1))=\frac{1}{2}(0+1)=\frac{1}{2}$ (1)+(1)

(4) a) Hom. Dirichlet BC + int. by part $\Rightarrow V=H_0^1(0,1)$ and (1)

$a(u,v)=4\int_0^1 u'(x)v'(x)\,dx+\int_0^1(x-1/2)u'(x)v(x)\,dx+5\int_0^1 u(x)v(x)\,dx$ (1)

and $\ell(v)=\int_0^1 f(x)v(x)\,dx$

b) Let $v\in V$, one has $a(v,v)=4\int_0^1(v'(x))^2\,dx+\int_0^1\underbrace{(x-1/2)}_{\le 1\ \forall x\in(0,1)}v'(x)v(x)\,dx+5\int_0^1 v$

Obs $|AB|\le\frac{1}{2}(A^2+B^2)$ hence $AB\ge-\frac{1}{2}(A^2+B^2)$

Thus $a(v,v)\ge 4\int(v'(x))^2\,dx-\frac{1}{4}\int_0^1((v'(x))^2+v(x)^2)\,dx+5\int_0^1 v(x)^2\,dx$

$\ge 3\int_0^1((v'(x))^2+v(x)^2)\,dx=3\|v\|_{H^1}^2$ ok (2)

c) $\ell(v)$ is linear and cont. $\rightsquigarrow$ ok (lecture) (1)

a bilinear (ok) and cont. ($|a(u,v)|\overset{C-S}{\le}4\|u'\|_{L^2}\|v'\|_{L^2}+$

$\frac{1}{2}\|u'\|_{L^2}\|v\|_{L^2}+5\|u\|_{L^2}\|v\|_{L^2}\le$ (1)

$\le 10\cdot\|u\|_{H^1}\|v\|_{H^1}$)

and b) $\Rightarrow$ L-M ok!

(5) a) $\begin{cases} w_t-w_{xx}=0 \\ w(0,t)=0=w(1,t) \\ w(x,0)=0 \end{cases}$ by linearity (1)

b) $\|w(\cdot,t)\|_{L^2}\le\|0\|_{L^2}+\int_0^t\|0\|_{L^2}\,ds=0$ ok (1)

(6) a) $V_h = \{ v \in \mathcal{C}^0(\bar\Omega) : v_{|T_i} \in \mathbb{P}^1 \text{ for } i=1,2,3,4 \text{ and } v=0 \text{ on } \Gamma_0 \}$ ①

$\mathbb{P}^1$ ↪ polyn. of degree ≤ 1 ①

b) Elements in V_h are determined by ~~their~~ values of hat fct at nodes A, B, C, D, E. ①

One also has $v_h(C) = v_h(D) = 0$ for $v_h \in V_h$ ①

⇝ only needs test fct on A, B, E and

$u_h = u_h(A)\varphi_A + u_h(B)\varphi_B + u_h(E)\cdot\varphi_E$ (*) ①

c) One solves $a(u_h, v_h) = \ell(v_h)$ for $v_h = \varphi_A, \varphi_B, \varphi_E$. ①

This gives $M\xi = b$, where $\xi = \begin{pmatrix} u_h(A) \\ u_h(B) \\ u_h(E) \end{pmatrix}$, $b = \begin{pmatrix} \int_\Omega \varphi_A \,dx\,dy \\ \int \varphi_B \\ \int \varphi_E \end{pmatrix}$ ①

and $M = \begin{pmatrix} \int_\Omega |\nabla\varphi_A|^2 & \int \nabla\varphi_A\cdot\nabla\varphi_B & \int \nabla\varphi_A \nabla\varphi_E \\ * & \int |\nabla\varphi_B|^2 & \int \nabla\varphi_B \nabla\varphi_E \\ * & * & \int |\nabla\varphi_E|^2 \end{pmatrix}$ ①

(symmetry)

Since $\varphi_A = \begin{cases} 1 & \text{on } A \\ 0 & \text{on } B, C, D, E \end{cases}$ ⇝ $\varphi_A(x,y) = \begin{cases} 0 & \text{for } (x,y) \in T_1 \cup T_4 \\ -x & \text{for } (x,y) \in T_2 \cup T_3 \end{cases}$ ①

etc.

d) $M_{11} = \int_\Omega |\nabla\varphi_A|^2 \,dx\,dy = \dots = \text{area}(T_2 \cup T_3) = 1$ ①

$M_{22} = \int_\Omega |\nabla\varphi_B|^2 \,dx\,dy = \dots = \text{area}(T_3 \cup T_4) = 1$ ①

<u>or</u> use ref. Δ from lecture.

(7) a) For $v \in V_h \subset H^1(I)$, one has $a(u,v) = (f,v) = a(u_h, v) \Rightarrow$ ①

$a(u_h - u, v) = 0$

b) $\alpha \|\Pi_h u - u_h\|_{H^1}^2 \underset{*}{\le} |a(\Pi_h u - u, \Pi_h u - u_h)| \underset{\text{cont.}}{\le} C \|\Pi_h u - u\|_{H^1} \cdot \|\Pi_h u - u_h\|_{H^1}$

* $a(\Pi_h u - u, \Pi_h u - u_h) = a(\Pi_h u - u_h, \Pi_h u - u_h)$ ① + coercivity ①

Hence, $\|\Pi_h u - u_h\|_{H^1}^2 \le \frac{C}{\alpha} \|\Pi_h u - u\|_{H^1} \cdot \|\Pi_h u - u_h\|_{H^1}$ Ok.

c) $\|u - u_h\|_{H^1} \underset{\Delta}{\le} \|u - \Pi_h u\|_{H^1} + \|\Pi_h u - u_h\|_{H^1} \underset{b)}{\le} \left(\frac{C}{\alpha} + 1\right) \cdot \|\Pi_h u - u\|_{H^1}$ Ok ①