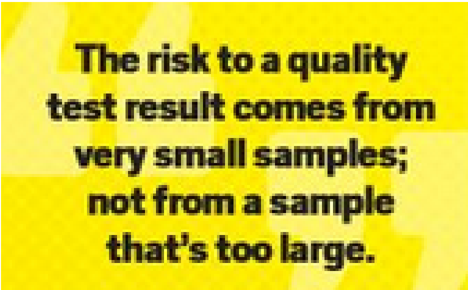
Slides 11: t-tests

- t-distributions
- Exact confidence interval for μ
- Exact confidence interval for σ
- One sample t-test
- Two sample t-test



In this special case, when a random sample $(x_1, \ldots, x_n)$ is taken from a normal distribution $\mathrm{N}(\mu, \sigma)$,

$$\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim t_{n-1}$$

has the so-called t-distribution with $n-1$ degrees of freedom. This implies an exact $100(1-\alpha)\%$ confidence interval

$$I_\mu = \bar{x} \pm t_{n-1}\left(\tfrac{\alpha}{2}\right) \cdot \frac{s}{\sqrt{n}}$$

For example,

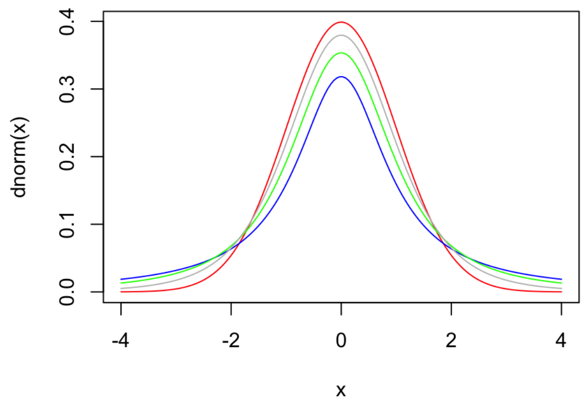
$t_{10}(0.025) = 2.23$,

$t_{20}(0.025) = 2.09$,

$t_{30}(0.025) = 2.04$

A t_k-distribution curve looks similar to N(0,1)-curve being symmetric around zero.

If $k \geq 3$, then the variance is $\frac{k}{k-2}$.



t-distribution curves with df $= 1, 2, 5, \infty$

If $Z, Z_1, \ldots, Z_k$ are N(0,1) and independent, then

$$\frac{Z}{\sqrt{(Z_1^2+\ldots+Z_k^2)/k}} \sim t_k.$$

Moreover, in the N(μ, σ) case we get access to an exact confidence interval formula for the variance thanks to the following result.

$$\boxed{\text{Exact distribution } \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}}$$

Exact $100(1-\alpha)\%$ confidence interval

$$I_\sigma = \Big(c(\tfrac{\alpha}{2})s,\ c(1-\tfrac{\alpha}{2})s\Big)$$

where $c^2(p) = \frac{n-1}{\chi^2_{n-1}(p)}$. Examples of 95% confidence intervals

$$I_\sigma = (0.69s, 1.82s) \text{ for } n = 10,$$
$$I_\sigma = (0.74s, 1.55s) \text{ for } n = 16,$$
$$I_\sigma = (0.78s, 1.39s) \text{ for } n = 25,$$
$$I_\sigma = (0.85s, 1.22s) \text{ for } n = 60.$$

For the normal model, $\mathrm{Var}(S^2) = \frac{2\sigma^4}{n-1}$, standard error for s^2 is $\sqrt{\frac{2}{n-1}}s^2$.

We wish to test H_0: $\mu = \mu_0$ against either the two-sided or a one-sided alternative. One-sample t-test is used for small n, under the assumption that the population distribution is normal. The t-test statistic

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}, \qquad T \overset{H_0}{\sim} t_{n-1}.$$

Example: Smoking and platelet aggregation

$n = 11$ paired observations (x_i, y_i) before and after smoking

$(25, 27); (25, 29); (27, 37); (28, 43); (30, 46); (44, 56); (52, 61); (53, 57); (53, 80); (60, 59); (67, 82)$

Assuming that the population distribution for the differences $d_i = x_i - y_i$ is normal , we test

$$H_0\colon \mu_1 - \mu_2 = 0 \text{ against } H_1\colon \mu_1 - \mu_2 \neq 0.$$

using the one-sample t-test. The observed test statistic value

$$t_{\text{obs}} = \frac{\bar{d}}{s_d/\sqrt{n}} = \frac{10.27}{2.40} = 4.28$$

gives two-sided p-value $2 * (1 - \text{pt}(4.28, 10)) = 0.0016$.

Two independent random samples $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_m)$ from two populations. The key assumption for the two-sample t-test:

Two normal population distributions $X \sim \mathrm{N}(\mu_1, \sigma)$, $Y \sim \mathrm{N}(\mu_2, \sigma)$.

In other words, there is a two level main factor plus noise $\mathrm{N}(0, \sigma)$. The two levels of the main factor are quantified by μ_1 and μ_2.

Define the pooled sample variance by

$$s_{\mathrm{p}}^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2 + \sum_{i=1}^{m}(y_i - \bar{y})^2}{n + m - 2}$$

Note that

$$s_{\mathrm{p}}^2 = \tfrac{n-1}{n+m-2} \cdot s_1^2 + \tfrac{m-1}{n+m-2} \cdot s_2^2$$

The pooled sample variance is an unbiased estimate of the variance σ^2:

$$\mathrm{E}(S_{\mathrm{p}}^2) = \tfrac{n-1}{n+m-2}\mathrm{E}(S_1^2) + \tfrac{m-1}{n+m-2}\mathrm{E}(S_2^2) = \sigma^2.$$

Question. There are $n + m$ terms in the numerator of s_{p}^2 but the denominator is $n + m - 2$. Why?

In the case of equal variances,

$$\mathrm{Var}(\bar{X} - \bar{Y}) = \mathrm{Var}(\bar{X}) + \mathrm{Var}(\bar{Y}) = \frac{\sigma^2}{n} + \frac{\sigma^2}{m} = \sigma^2 \frac{n+m}{nm},$$

which yields the following expression for the standard error

$$s_{\bar{x}-\bar{y}} = s_p \sqrt{\frac{n+m}{nm}}.$$

$$\boxed{\text{Exact distribution } \frac{(\bar{X}-\bar{Y})-(\mu_1-\mu_2)}{S_{\mathrm{p}}} \cdot \sqrt{\frac{nm}{n+m}} \sim t_{n+m-2}}$$

Exact confidence interval formula

$$I_{\mu_1-\mu_2} = \bar{x} - \bar{y} \pm t_{n+m-2}(\tfrac{\alpha}{2}) \cdot s_{\mathrm{p}} \cdot \sqrt{\frac{n+m}{nm}}.$$

Two sample t-test uses the test statistic $t = \frac{\bar{x}-\bar{y}}{s_{\mathrm{p}}} \cdot \sqrt{\frac{nm}{n+m}}$ for testing H_0: $\mu_1 = \mu_2$. The null distribution of the test statistic is

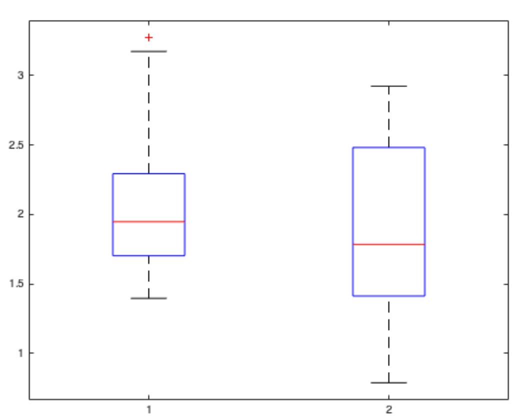
$$T \sim t_{n+m-2}$$

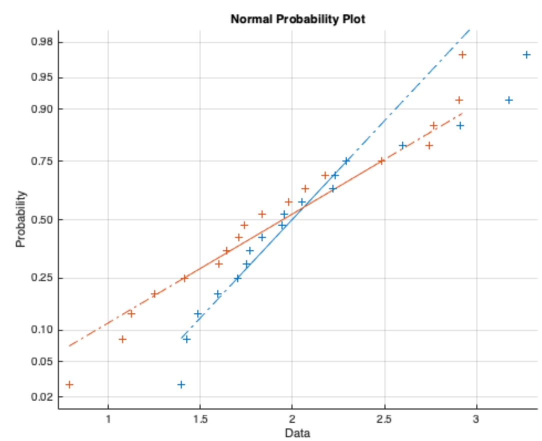
The data on percentage of iron retained by mice

$$\mathrm{Fe}^{2+}:\ n = 18,\ \bar{x} = 9.63,\ s_1 = 6.69,\ s_{\bar{x}} = 1.58$$
$$\mathrm{Fe}^{3+}:\ m = 18,\ \bar{y} = 8.20,\ s_2 = 5.45,\ s_{\bar{y}} = 1.28$$

has "iron form" as the main factor with two levels Fe^{2+} and Fe^{3+}. The boxplots and normal probability plots show that the distributions are not normal. After the log transformation the data look more like normally distributed

$$\bar{x}' = 2.09,\ s_1' = 0.659,\ s_{\bar{x}'} = 0.155,$$
$$\bar{y}' = 1.90,\ s_2' = 0.574,\ s_{\bar{y}'} = 0.135.$$





For the log-transformed data we get $t_{\mathrm{obs}} = 0.917$, $\mathrm{df} = 34$, so that the two-sided p-value $= 36.6\%$.