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A Very Brief Review of Signals and Systems

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1 Introduction

This document should be viewed as a very brief review to some key signal and systems concepts. Many details are omitted and some mathematical rigor has been neglected to shorten the presentation (but, hopefully, without losing relevance). For an in-depth discussion of the material, we recommend the excellent book by Oppenheim and Willsky [1].

2 Continuous-Time and Discrete-Time Signals

We classify signals as being either *continuous-time* (functions of a real-valued variable) or *discrete-time* (functions of an integer-valued variable). To signify the difference, we (usually) use round parenthesis around the argument for continuous-time signals, e.g., $x(t)$ and square brackets for discrete-time signals, e.g., $x[n]$. We will also use the notation x_n for discrete-time signals.

One possible way to create a discrete-time signal, $x_d[n]$, is by sampling a continuous-time signal, $x_c(t)$, every T_s seconds:

$$x_d[n] = x_c(nT_s), \qquad \text{for } n = 0, \pm 1, \pm 2, \dots$$

We refer to T_s and $f_s = 1/T_s$ as the sample interval and sample frequency, respectively.

We will assume that signals are complex-valued if not explicitly stated otherwise.

3 Useful Signals

The *unit step function* is defined as

$$u(t) \triangleq \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}.$$

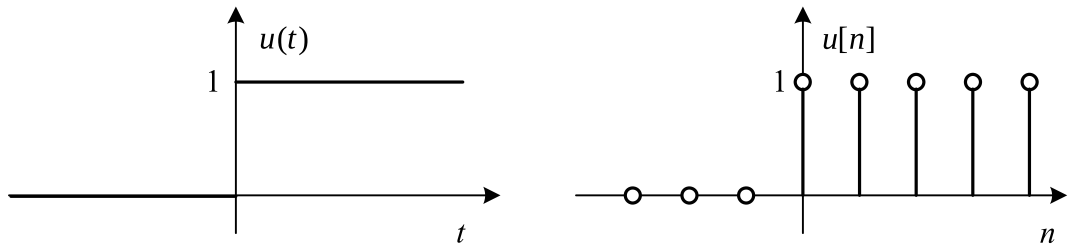

Figure 1: Unit step functions

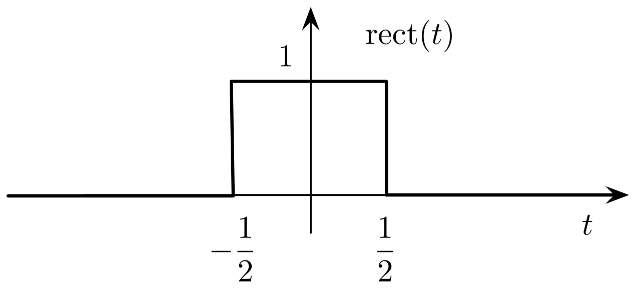

Figure 2: Rectangular function

We will leave the value of $u(t)$ at $t = 0$ undefined for the time being. Readers that feel uneasy about this can use $u(0) = 1/2$.

The discrete-time unit step function is defined as

$$u[n] \triangleq \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}.$$

The signal $\mathrm{rect}(t)$ is a square pulse of unit width

$$\mathrm{rect}(t) \triangleq \begin{cases} 1, & |t| < \frac{1}{2} \\ 0, & |t| > \frac{1}{2} \end{cases}.$$

We will leave the value of $\mathrm{rect}(t)$ at $t = \pm 1/2$ undefined. Following the pattern from the unit step function, we could use $\mathrm{rect}(t) = 1/2$ for $t = \pm 1/2$.

The continuous-time impulse (or *Dirac delta function*) is a strange signal. To make a long story short, we will think of the Dirac delta function as the limit of the function $\delta_\varepsilon(t)$, which is defined as

$$\delta_\varepsilon \triangleq \begin{cases} \frac{1}{\varepsilon}, & |t| < \frac{\varepsilon}{2} \\ 0, & |t| > \frac{\varepsilon}{2} \end{cases}.$$

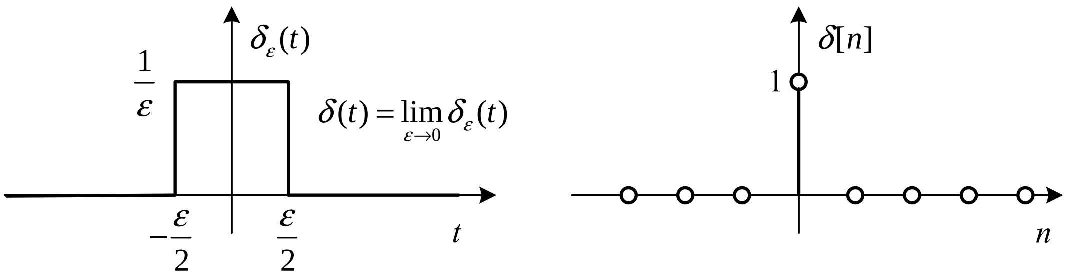

Figure 3: Delta functions

Hence, as seen in Figure 3, $\delta_\varepsilon(t)$ is a square pulse of height $1/\varepsilon$ and width ε. The Dirac delta function, $\delta(t)$, can be thought of as the limit of $\delta_\varepsilon(t)$ when $\varepsilon \to 0$. We can therefore think of the Dirac delta function as an infinitely high and infinitely thin square pulse. Such a signal has the peculiar property that for any function $f(x)$ (assumed to be continuous at $x = 0$),

$$\int_{-\infty}^{\infty} f(x)\delta(x)\,dx = f(0).$$

As a matter of fact, the above equation is often taken as the definition of $\delta(t)$.

The discrete-time impulse (or *Kronecker delta function*) is an ordinary signal defined as

$$\delta[n] \triangleq \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases}.$$

4 Time-Shift, Time-Compression, and Folding

We will frequently need to delay or advance signals in time. Mathematically, this is achieved by substitution of the time variable. For instance, if we want $y(t)$ to be equal to $x(t)$ delayed with T seconds, the appropriate substitution is

$$y(t) = x(t)|_{t \to t-T} = x(t - T).$$

Conversely, if we want $z(t)$ to be equal to $x(t)$ advanced in time with T seconds, then

$$z(t) = x(t)|_{t \to t+T} = x(t + T).$$

We will also find it useful to fold signals, i.e., to reflect or mirror the signal around the point $t = 0$ on the time-axis. This is done by the substitution $t \to -t$. Hence,

$$w(t) = x(t)|_{t \to -t} = x(-t),$$

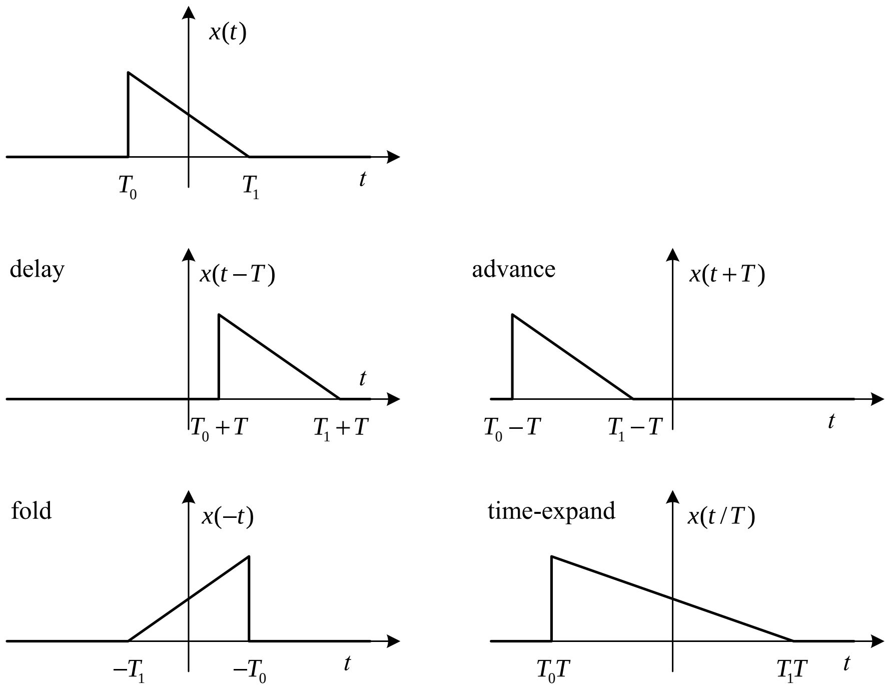

Figure 4: Delayed, advanced, folded and time-expanded versions of $x(t)$.

Substitution	Interpretation
$t \to t - T$	delay (right-shift) with T seconds
$t \to t + T$	advance (left-shift) with T seconds
$t \to -t$	fold (mirror) around the point $t = 0$
$t \to t/T$	expansion with a factor T

Table 1: Fundamental substitutions of the time-variable

is a folded version of $x(t)$.

Another substitution of interest is $t \to t/T$, which corresponds to time-expanding the signal by a factor of T. For instance, the signal $\mathrm{rect}(t/T)$ is a rectangular pulse of width T seconds. Of course, we can choose $T < 1$, and then the pulse $\mathrm{rect}(t/T)$ will be thinner than $\mathrm{rect}(t)$.

A signal $x(t)$ and its transformations is found in Figure 4. A summary of the fundamental substitutions is found in Table 1.

The fundamental substitutions can be combined to obtain more complicated transformations. For instance, if we want to fold a signal $x(t)$ and delay it by T

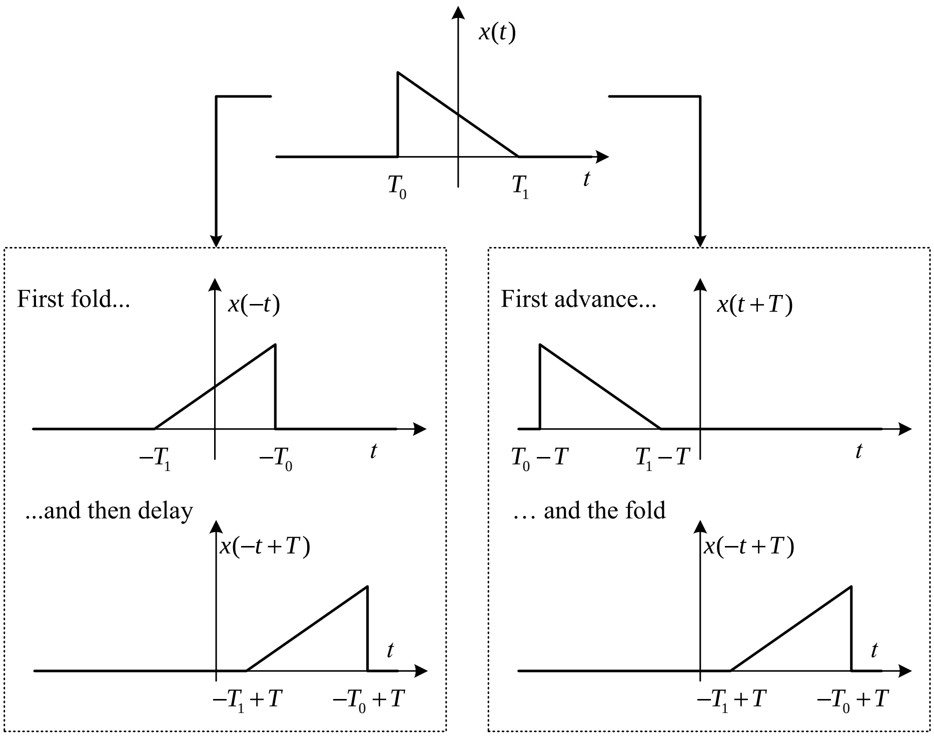

Figure 5: Two ways to arrive at $x(-t+T)$.

seconds, then this can be done as

$$x(t) \xrightarrow[\text{fold}]{t\to -t} x(-t) \xrightarrow[\text{delay}]{t\to t-T} x(-(t-T)) = x(T-t).$$

If we think about it for a while, we realize that we can achieve the same result by first *advancing* the signal by T seconds and then folding it:

$$x(t) \xrightarrow[\text{advance}]{t\to t+T} x(t+T) \xrightarrow[\text{fold}]{t\to -t} x(-t+T)) = x(T-t).$$

Please note that a common mistake is to believe that the folded version of $x(t+T)$ is $x(-(t+T)) = x(-t-T)$, which obviously is wrong. The process is illustrated in Figure 5.

4.1 Signal Energy and Signal Power

We will find it useful to characterize the “size” of a signal. One such measure of “size” is the signal energy. The signal energy, or just energy, of the signal $x(t)$ is

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2\, dt. \tag{1}$$

When the integral does not converge, we say that the signal has infinite energy. A signal that is time-limited and has finite amplitude ($|x(t)| < \infty$) has finite energy[1]. Since all real communication signals are time-limited and have finite amplitudes, this means that all real communication signals have finite energy. However, for analysis and design of real, physical systems, it is also meaningful and convenient to use non-physical signals and systems.

An example of a class of very useful (but non-physical) signals are the periodic signals, in particular complex exponentials and sinusoids. It is clear from (1) that all nonzero periodic signals have infinite energies. Hence, to say something meaningful about the "size" of these signals, we cannot use energy. However, many periodic signals have finite (signal) power. For a general signal, periodic or aperiodic, the power is defined as

$$P_x = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt. \tag{2}$$

For a periodic signal $x_p(t)$ with period T_p, i.e., a signal with the property $x_p(t) = x_p(t + T_p)$ for all t, the power is simply the energy for one period of the signal divided by the period,

$$P_{x_p} = \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} |x_p(t)|^2 \, dt.$$

5 Systems

A system is an entity that takes an input signal and produces an output signal. Systems can be linear or nonlinear and time-invariant or time-varying.

5.1 Linear Systems

A *linear* system follows the *superposition principle.* The superposition principle says that the output to a linear combination of input signals is the same linear combination of the corresponding output signals. To be precise, if the input signals $x_1(t)$ and $x_2(t)$ corresponds to the output signals $y_1(t)$ and $y_2(t)$, respectively, then the input signal $a_1x_1(t) + a_2x_2(t)$ should correspond to the output signal $a_1y_1(t) + a_2y_2(t)$ for any constants a_1 and a_2.

5.2 Time-Invariant Systems

In words, a *time-invariant system* is a system which does not change with time. Mathematically, if the input $x(t)$ gives the output $y(t)$, then the system is time-

[1]We will implicitly assume that signals are Riemann-integrable, i.e., that signal is bounded and continuous almost everywhere. This is reasonable for most, if not all, physical signals.

invariant if the input $x(t-T)$ gives the output $y(t-T)$ for any delay T. Hence, a time-shift of the input gives the same time-shift of the output.

5.3 Linear Time-Invariant Systems

Linear time-invariant (LTI) systems are commonly used in signal processing systems. LTI systems have many nice properties, e.g., an LTI system is completely described by its *impulse response*. The impulse response is the output when the input is an impulse, i.e., a Dirac delta function for continuous-time systems or a Kronecker delta function for discrete-time systems.

It is not difficult to show that if $h(t)$ is the impulse response of an LTI system, then the output to the input $x(t)$ can be found as

$$y(t) = \int_{-\infty}^{\infty} x(u)h(t-u)\,du = x(t) * h(t). \tag{3}$$

This integral is known as a *convolution* integral and $*$ is used to denote convolution. To visualize the integrand $f(u) = x(u)h(t-u)$, it is important to remember that the integrand should be considered as function of u (the integration variable). Hence, $h(t-u)$ should be thought of as being $h(u)$ folded and delayed by t seconds (or, equivalently, $h(u)$ advanced by t seconds and then folded). See Figure 6 for an example. With a bit of practice, it will be easy to visualize the integrand, which is really helpful to understand convolution. For instance, we realize that for each new value of t, we will shift $h(-u+t)$ to a new position on the u-axis. The integrand and integral will therefore (probably) change.

Similarly, the output of a discrete-time LTI system with impulse response $h[n]$ and input $x[n]$ is the convolution between $x[n]$ and $h[n]$,

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n].$$

As seen, convolution is an operation on two signals which results in a third signal. Convolution is *commutative,*

$$x(t) * h(t) = h(t) * x(t),$$

associative,

$$[x(t) * y(t)] * z(t) = x(t) * [y(t) * z(t)],$$

and *distributive,*

$$x(t) * [y(t) + z(t)] = x(t) * y(t) + x(t) * z(t).$$

These properties applies for discrete-time convolution as well.

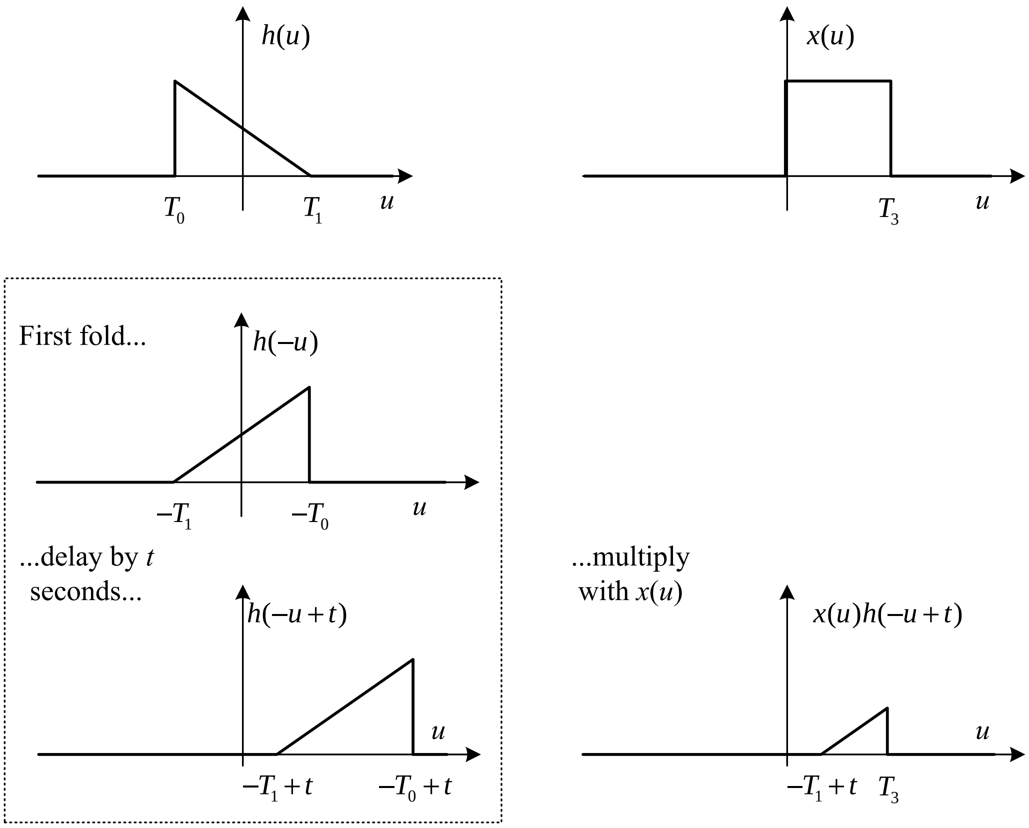

Figure 6: An example how to compute $x(u)h(t-u)$, i.e., the integrand of the convolution integral.

5.4 Correlation and Convolution

The *crosscorrelation function* between $x(t)$ and $y(t)$ is defined as

$$R_{xy}(t) = \int_{-\infty}^{\infty} x(t+u)y^*(u)\, du,$$

where $y^*(t)$ is the complex conjugate of $y(t)$. When just speaking about the *correlation* between $x(t)$ and $y(t)$, we (usually) mean $R_{xy}(0)$.

We note that the crosscorrelation function is quite related with convolution. In fact, we can compute the crosscorrelation function above by processing $x(t)$ with an LTI system with impulse response $h(t) = y^*(-t)$. Such as system is called a filter that is *matched* to $y(t)$. To show that the output is actually $R_{xy}(t)$, we recall that the filter output, $z(t)$, is the convolution between $x(t)$ and $h(t)$,

$$\begin{aligned} z(t) &= x(t) * h(t) \\ &= \int_{-\infty}^{\infty} x(u) \underbrace{h(t-u)}_{=y^*(-(t-u))} du \\ &= \int_{-\infty}^{\infty} x(u)y^*(u-t)\,du \\ &= \int_{-\infty}^{\infty} x(v+t)y^*(v)\,dv \\ &= R_{xy}(t). \end{aligned}$$

A short-hand notation for the above relation is

$$R_{xy}(t) = x(t) * y^*(-t).$$

The connection between the crosscorrelation function and convolution is quite useful for understanding how digital communication receivers can be implemented.

The *autocorrelation function* of $x(t)$ is $R_{xx}(t)$, i.e., the crosscorrelation of $x(t)$ with itself,

$$R_{xx}(t) = \int_{-\infty}^{\infty} x(t+u)x^*(u)\,du.$$

An interesting property of the autocorrelation function is that $R_{xx}(0) = E_x$, where E_x is the energy of $x(t)$,

$$E_x = \int_{-\infty}^{\infty} |x(u)|^2\,du.$$

Another very useful property is that $|R_{xx}(t)|$ attains its maximal value E_x when $t = 0$. To prove this, we note that, from the Schwarz inequality[2],

$$|R_{xx}(t)|^2 = \left| \int_{-\infty}^{\infty} x(t+u)x^*(u)\,du \right|^2 \le \int_{-\infty}^{\infty} |x(t+u)|^2\,du \int_{-\infty}^{\infty} |x^*(u)|^2\,du = E_x^2,$$

where the inequality hold with equality if and only if $x(t+u) = \alpha x(u)$ for some complex constant α. Hence, $|R_{xx}(t)| \le E_x$ for all values of t, but since $R_{xx}(0) = E_x$, this proves that $|R_{xx}(t)|$ attains its maximal value (E_x) when $t = 0$.

This fact is very useful for understanding how digital communication receivers can be synchronized.

[2]The Schwarz inequality (also known as the Cauchy-Schwarz inequality) says that if $g(t)$ and $h(t)$ have finite energies, then

$$\left| \int_{-\infty}^{\infty} g(t)h(t)\,dt \right|^2 \le \int_{-\infty}^{\infty} |g(t)|^2\,dt \int_{-\infty}^{\infty} |h(t)|^2\,dt$$

with equality if and only if $g(t) = \alpha h^*(t)$, where α is a complex constant.

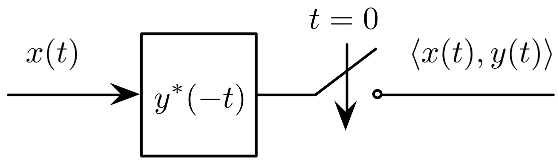

Figure 7: Calculating inner product by sampling a linear filter

5.5 Inner Product and Norm

The inner product between two signals $x(t)$ and $y(t)$ is defined as

$$\langle x(t), y(t)\rangle = \int_{-\infty}^{\infty} x(t)y^*(t)\, dt,$$

and the norm of the signal $x(t)$ is defined as

$$\|x(t)\| = \sqrt{\langle x(t), x(t)\rangle} = \sqrt{\int_{-\infty}^{\infty} |x(t)|^2\, dt} = \sqrt{E_x}.$$

The concept of inner products and norms is perhaps familiar from previous exposure to linear algebra. Indeed, the inner product is also known as the scalar product between two vectors and the norm as the length of a vector. It can actually be shown that the set of all signals with finite energy forms a linear vector space (just as the set of all points in three-dimensional space forms a vector space). We will therefore say that the length of a signal $x(t)$ is $\|x(t)\|$ and the distance between $x(t)$ and $y(t)$ is $\|x(t) - y(t)\|$.

From the defintion of inner product, it is easy to show that

$$\begin{aligned}
\langle x(t), y(t)\rangle &= [\langle y(t), x(t)\rangle]^* && (4)\\
\langle ax(t), y(t)\rangle &= a\langle x(t), y(t)\rangle && (5)\\
\langle x(t), ay(t)\rangle &= a^*\langle x(t), y(t)\rangle && (6)\\
\langle x_1(t) + x_2(t), y(t)\rangle &= \langle x_1(t), y(t)\rangle + \langle x_2(t), y(t)\rangle && (7)
\end{aligned}$$

where a is a complex number.

From the definition of the crosscorrelation function, we see that $\langle x(t), y(t)\rangle = R_{xy}(0)$, or

$$\langle x(t), y(t)\rangle = R_{xy}(t)|_{t=0} = x(t) * y^*(-t)|_{t=0}.$$

Hence, we can calculate inner products by sampling the output of a linear filter, see Fig. 7.

5.6 Frequency Domain Representations of Signals and Systems

We use Fourier transforms and Fourier series to represent signals as linear combinations of complex exponentials. A continuous-time complex exponential with amplitude A, frequency f_0, and phase ϕ (the quantities A, f_0, and ϕ are real numbers) can be written as

$$z(t) = Ae^{j(2\pi f_0 t+\phi)} = A\cos(2\pi f_0 t + \phi) + jA\sin(2\pi f_0 t + \phi).$$

Obviously, the complex exponential is a complex-valued function which is periodic with fundamental period $1/f_0$. The signal $z(t)$ represents a single-frequency component at the frequency f_0.

If $z(t)$ is the input to an LTI system with impulse response $h(t)$, then the output is

$$\begin{aligned} y(t) &= h(t) * z(t) \\ &= \int_{-\infty}^{\infty} h(u)z(t-u)\,du \\ &= \int_{-\infty}^{\infty} h(u)e^{j(2\pi f_0(t-u)+\phi)}\,du \\ &= \underbrace{e^{j(2\pi f_0 t+\phi)}}_{=z(t)} \underbrace{\int_{-\infty}^{\infty} h(u)e^{-j2\pi f_0 u}\,du}_{=H(f_0)} \\ &= z(t)H(f_0) \end{aligned}$$

where $H(f)$ is the Fourier transform of the impulse response, also known as the *frequency response* of the system,

$$H(f) = \int_{-\infty}^{\infty} h(t)e^{-j2\pi f t}\,dt.$$

We conclude that the output to a complex exponential is the same exponential scaled with the frequency response of the system.

By invoking the inverse Fourier transform, we can write $h(t)$ as

$$h(t) = \int_{-\infty}^{\infty} H(f)e^{j2\pi f t}\,df.$$

Hence, $h(t)$ can be seen as a sum of many complex exponentials (frequency components), and the frequency component at frequency f has complex amplitude $H(f)$. Hence, $H(f)$ represents the signal's distribution in frequency.

Now, for an input signal $x(t)$ with Fourier transform $X(f)$, the output to an LTI system with impulse response $h(t)$ and frequency response $H(f)$ is $y(t) = h(t) * x(t)$. The Fourier transform of $y(t)$ can be shown to be

$$Y(f) = H(f)X(f).$$

5.7 Bandlimited Signals and the Sampling Theorem

We can compute the energy of a signal in the frequency domain by using Parseval's theorem

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X(f)|^2 \, df,$$

where $X(f)$ is the Fourier transform of $x(t)$. From the above equation, we see that the signals total energy can be found by integrating the energy distribution in time, $|x(t)|^2$, or the energy distribution in frequency, $|X(f)|^2$. For this reason, the quantity $|X(f)|^2$ is called the signal's energy density function.

A signal $x(t)$ is said to have bandwidth W if $x(t)$ has negligible energy outside the frequency band $-W \leq f \leq W$. There are many definitions of bandwidth, each corresponding to a certain notion of what is considered negligible. The most straight-forward definition is called absolut bandwidth, which requires that $|X(f)|^2 = 0$ for $|f| > W$. The condition is equivalent with $X(f) = 0$ for $|f| > W$.

We can create an absolute bandlimited signal from a general signal $x(t)$ by filtering with an ideal lowpass filter. An ideal lowpass filter has rectangular shaped frequency response

$$H_{LP}(f) = \text{rect}\left(\frac{f}{2W}\right) = \begin{cases} 1, & |f| \geq W \\ 0, & \text{otherwise} \end{cases},$$

where W is known as the cutoff frequency. Cleary, the output $y(t)$ will have a bandlimited spectrum $Y(f) = H(f)X(f) = 0$ for $|f| > W$.

The impulse response of an ideal lowpass filter is easily calculated as the inverse Fourier transform of the frequency response

$$\begin{aligned} h_{LP}(t) &= \int_{-\infty}^{\infty} H_{LP}(f) e^{j2\pi f t} \, df \\ &= \int_{-W}^{W} e^{j2\pi f t} \, df \\ &= \frac{1}{j2\pi t}(e^{j2\pi W t} - e^{-j2\pi W t}) \\ &= \frac{1}{\pi t} \sin(2\pi W t) \\ &= 2W \, \text{sinc}(2Wt) \end{aligned}$$

where the sinc function is defined as

$$\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t},$$

where we use the convention that $\text{sinc}(0) = 1$. The above definition is the most widely used one, but we should note that some authors define $\text{sinc}(t) = \sin(t)/t$.

The sinc pulse plays a key role in conversion between discrete-time and continuous-time signals. The sampling theorem states that we can construct a bandlimited signal from its samples. Consider the signal $y(t)$ that we found by filtering $x(t)$ with an ideal lowpass filter with cutoff frequency W. The sampling theorem states that we can construct $y(t)$ from its samples $y[n] = y(nT_s)$ as

$$y(t) = \sum_{n=-\infty}^{\infty} y[n] \operatorname{sinc}\left(\frac{t - nT_s}{T_s}\right) \tag{8}$$

if the sample frequency $f_s = 1/T_s > 2W$.

6 Linear Combinations

We will frequently construct complicated signals as linear combinations of other, simpler signals. For example, if the signal $x(t)$ can be written as

$$x(t) = (-1)\operatorname{rect}(t) + 2\operatorname{rect}(t-1), \tag{9}$$

then we say that $x(t)$ is a linear combination of the signals $\operatorname{rect}(t)$ and $\operatorname{rect}(t-1)$. In general, we say that $x(t)$ is a linear combination of the signals in the set

$$\{\ldots, x_{-1}(t), x_0(t), x_1(t), \ldots\}$$

if we can write

$$x(t) = \sum_{k=-\infty}^{\infty} a_k x_k(t) \tag{10}$$

where a_k are some scalars (real or complex numbers) for all k. In the example (9) above,

$$a_k = \begin{cases} -1, & k = 0 \\ 2, & k = 1 \\ 0, & \text{otherwise} \end{cases}$$

and $x_1(t) = \operatorname{rect}(t)$ and $x_2(t) = \operatorname{rect}(t-1)$.

It is useful to think of the signal $x(t)$, where

$$x(t) = \int_{-\infty}^{\infty} a(u)x(u,t)\,du, \tag{11}$$

as a linear combination of the signals $x(u,t)$ with the "scalars" $a(u)$. For this to make sense, we must think of $x(u,t)$ as a signal, i.e., a function of t, not u. In other words, for each fixed value of u, say $u = u_0$, we have a signal $x(u_0,t)$ that varies in time t. For each new value of u, say $u = u_k$, we get a new signal $x(u_k,t)$. In the same way, we must think of $a(u)$ as a scalar for any fixed value of u.

To see the similarity between (11) and (10), let us approximate the integral in (11) by its left Riemann sum

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} a(u)x(u,t)\,du \\ &\approx \sum_{k=-\infty}^{\infty} a(k\Delta u)x(k\Delta u,t)\Delta u \\ &= \sum_{k=-\infty}^{\infty} \underbrace{a(k\Delta u)\Delta u}_{=a_k}\underbrace{x(k\Delta u,t)}_{=x_k(t)} = \sum_{k=-\infty}^{\infty} a_k x_k(t). \end{aligned}$$

Armed with this new insight, we see that the output $y(t)$ to an LTI system with impulse response $h(t)$ and input $x(t)$, i.e.,

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(u)h(t-u)\,du,$$

can be viewed as a linear combination of shifted impulse responses, $h(t-u)$. Alternatively, since

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(u)x(t-u)\,du,$$

the output is also a linear combination of shifted inputs, $x(t-u)$.

Moreover, a signal $x(t)$ with Fourier transform $X(f)$ can be written as

$$x(t) = \int_{-\infty}^{\infty} X(u)e^{j2\pi ut}\,du.$$

Hence, $x(t)$ can be viewed as a linear combination of the complex exponentials $\exp(j2\pi ut)$.

We can also write

$$x(t) = \int_{-\infty}^{\infty} x(u)\delta(t-u)\,du.$$

Hence, $x(t)$ can be viewed as a linear combination of the shifted delta functions $\delta(t-u)$.

As a final example, consider the reconstruction formula from the sampling theorem (8). If $x(t)$ is bandlimited to W Hz and the sample duration $T_s < 1/(2W)$, then

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\mathrm{sinc}\left(\frac{t-nT_s}{T_s}\right).$$

Hence, $x(t)$ is a linear combination of shifted and time-expanded sinc-pulses, $\mathrm{sinc}(t/T_s)$.

6.1 Pulse Amplitude Modulation (PAM)

Pulse amplitude modulation (PAM) constructs a continuous-time signals $x(t)$ as

$$x(t) = \sum_{k=-\infty}^{\infty} a_k p(t - kT). \tag{12}$$

Hence, $x(t)$ is a linear combination of time-shifted versions of the pulse shape $p(t)$. Indeed, we obtain (12) from the general formula for a linear combination (10) by letting $x_k(t) = p(t - kT)$.

In digital communications, PAM is used for converting data symbols to signals suitable for the transmission medium at hand, a process that is called modulation. In this context, the amplitudes a_k represent one or several bits that are to be transmitted over the channel.

6.2 Simulation of Continuous-Time Systems

It is common to simulate systems to verify designs and theoretical calculations. In a tool like MATLAB, it is very easy to perform discrete-time convolutions between time-limited signals. Suppose that we have the signals $x[n]$ and $h[n]$ stored in the MATLAB vectors `x` and `h`, respectively. Then the convolution $z[n] = h[n] * x[n]$ can be computed as

```
>> z = conv(h, x);
```

However, for continuous-time systems, we must in general use numeric integration to compute convolutions. If we want to convolve $x(t)$ with $h(t)$, then the integral that needs to be computed is

$$y(t) = \int_{-\infty}^{\infty} h(u)x(t-u)\,du.$$

We can approximately compute this integral as

$$\int_{-\infty}^{\infty} h(u)x(t-u)\,du \approx \sum_{k=-\infty}^{\infty} h(kT_s)x(t - kT_s)T_s,$$

where T_s is the step size or sample interval. If we compute samples of $y(t)$ with the same sample interval, we arrive at the equation

$$\begin{aligned} y(nT_s) &= \int_{-\infty}^{\infty} h(u)x(nT_s - u)\,du \\ &\approx \sum_{k=-\infty}^{\infty} h(kT_s)x(nT_s - kT_s)T_s \\ &= T_s \sum_{k=-\infty}^{\infty} h(kT_s)x([n-k]T_s). \end{aligned} \tag{13}$$

The summation in the last line is nothing else but the convolution between the discrete-time signals $h_d[n] = h(nT_s)$ and $x_d[n] = x(nT_s)$.

The quality of the approximation becomes better as T_s decreases. As a matter of fact, if $x(t)$ and $h(t)$ are bandlimited, e.g., if $X(f) = H(f) = 0$ for $|f| > W$, then the approximation error disappear if $T_s < 1/(2W)$. (This is a consequence of the sampling theorem, which is proved and discussed at length in any decent signals and system book.)

To compute (13) in practice, the signals must also be truncated to make the vectors have finite length. This obviously leads to approximation errors if the signals are not time-limited. However, if the main features of the signals remain after truncation, the approximation may still be of use.

To summarize, we can compute samples of $y(t) = h(t) * x(t)$ as

1. Choose a sample interval T_s that is sufficiently small.
2. If $x(t)$ and $h(t)$ are not time-limited, then truncate them such that the main part of each signal still remains.
3. Sample the signals and store them in MATLAB vectors `x` and `h`.
4. Compute samples of $y(t)$ as `y = T_s * conv(x, h);`.

7 References

[1] Alan V. Oppenheim and Alan S. Willsky. *Signals and Systems*. Prentice-Hall, Upper Saddle River, New Jersey 07458, USA, second edition, 1997.