MATHEMATICS
Univ. of Gothenburg and Chalmers University of Technology
Brief solutions to examination in algebra: MMG500 and MVE 150, 2019-06-10.

1a) False. If p is a prime, then $\mathbf{Z}_p[x]$ is an integral domain of characteristic p, but not a field.

1b) True. If $a\neq 0, b$ are elements in a field K, then $ab=0 \Rightarrow a^{-1}(ab)=0 \Rightarrow (a^{-1}a)b=0 \Rightarrow b=0$.

1c) False. $\mathbf{Z}_6$ is a finite commutative ring with $[2][3]=[0]$ in $\mathbf{Z}_6$.

2) There exists by Euclid´s algorithm integers a,b with $ma+nb=(m, n)$. We have, therefore, if $g^m, g^n \in H$ that $g^{(m, n)} = g^{ma+nb} = (g^m)^a(g^n)^b \in H$. So if m and n are coprime, then $g\in H$.

3) If all $I_m=\{0\}$, then there is nothing to prove. We may hence assume that $I_k \neq \{0\}$for some k. Then I_m is generated by a monic polynomial for all $m\geq k$ with f_{m+1} dividing f_m as $I_m \subseteq I_{m+1}$. We have therefore a sequence of non-negative integers $\deg f_k \geq \deg f_{k+1} \geq \ldots$,which will become stationary $\deg f_n = \deg f_{n+1} = \ldots$ after some $n\geq k$. As f_{m+1} divides f_m for all m we have thus that $f_n = f_{n+1} = \ldots$ and $I_n = I_{n+1} = \ldots$, as was to be proved.

4) There are 24 rotations of the cube. They are (see Example 57.3 in Durbin's book)

1. The identity.
2. Three 180° rotations around lines joining the centers of opposite faces.
3. Six 90° rotations around lines joining the centers of opposite faces.
4. Six 180° rotations around lines joining the midpoints of opposite edges.
5. Eight 120° rotations around lines joining opposite vertices.

These rotations form a group G acting on the set T of 3-colourings of the sides. For $g\in G$, let $\Psi(g)$ be the number of 3-colourings preserved by g. It is equal to $3^{n(g)}$ for the number $n(g)$ of orbits of the action of $\langle g\rangle$ on the set S of the six sides of the cube.

We have for $\underline{g}$ of type 1, 2, 3, 4 resp. 5 the following $\langle g\rangle$–orbits on S.

1. Six orbits of length 1.
2. Two orbits of length 1 and two orbits of length 2.
3. Two orbits of length 1 and one orbit of length 4.
4. Three orbits of length 2.
5. Two orbits of length 3.

In particular, $\Psi(g)=3^6$, 3^4 , 3^4 , 3^3 resp. 3^2 such that $o(G)^{-1}\sum_{g\in G}\Psi(g) = \frac{1}{24}\sum_{g\in G}3^{n(g)} =$

$\frac{1}{24}(1\times 3^6 + 3\times 3^4 + 6\times 3^3 + 6\times 3^3 + 8\times 3^2) = \frac{3^2}{24}(1\times 3^4 + 3\times 3^2 + 6\times 3^1 + 6\times 3^1 + 8\times 3^0) = \frac{3}{8}\, 152 = 57.$

There are thus by Burnside's lemma 57 inequivalent 3-colourings of the six sides.

5 See Durbin's book

6 See Durbin's book