MVE035/600 Exercise session 1.1

Wednesday, 20 January 2021 07:46

2.1: e) Beräkna de partiella första-derivatorna av

$$f(x,y) = \log \underbrace{\sqrt{x^2+y^2}}_{= g(x,y)}$$

OBS: "Betrakta y som konstant när vi deriverar m.a.p. x"

$$\frac{\partial}{\partial x} f(x,y) = \frac{d(\log)}{dg} \cdot \frac{\partial g}{\partial x}(x,y)$$

"kedjeregel" $\left(= \frac{d(\sqrt{\cdot})}{d(x^2+y^2)} \cdot \frac{\partial(x^2+y^2)}{\partial x} = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}} \right)$

$$= \frac{1}{g(x,y)} \cdot \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{x^2+y^2}$$

$$\frac{\partial}{\partial y} f(x,y) = \{ f(x,y) = f(y,x) \} = \frac{y}{x^2+y^2}$$

I allmänhet: $f(x) = \log \|x\| \quad (= F(\|x\|))$

$$\frac{\partial}{\partial x_i} \|x\| = \frac{x_i}{\|x\|}$$

$$\Longrightarrow \frac{\partial f}{\partial x_i}(x) = \frac{1}{\|x\|} \cdot \frac{x_i}{\|x\|} = \frac{x_i}{\|x\|^2}$$

$$\left(= \frac{x_i}{\|x\|} F'(\|x\|) \right)$$

2.2: b) Beräkna $\frac{\partial f}{\partial x_k}$ $\forall k$ för

$$f(x) = \|x\| \quad (i\ \mathbb{R}^3)$$

$\partial \quad (\sum^n x^2)^{1/2}$

$$f(x) = \Big(\sum_{i=1} x_i \Big|$$

$$\frac{\partial f}{\partial x_k}(x) = \frac{1}{\cancel{2}\left(\sum_{i=1}^{n} x_i^2\right)^{1/2}} \cdot \cancel{2} x_k = \frac{x_k}{\left(\sum_{i=1}^{n} x_i^2\right)^{1/2}} = \frac{x_k}{\|x\|}$$

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2.8. c) Visa att

$$f(x,y) = e^{x+2y}, \quad (x,y) \in \mathbb{R}^2$$

är differentierbar i $(2,2)$.

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f är diff.bar i $(2,2)$ om $\exists A \in \mathbb{R}, B \in \mathbb{R}$ s.a.

$$f(2+h, 2+k) - f(2,2) = Ah + Bk + \|(h,k)\| \; o(\|(h,k)\|).$$

1) $$f(2+h, 2) - f(2,2) = e^{2+h+4} - e^{2+4}$$
$$= e^6(e^h - 1)$$
$$\overset{\text{Taylor}}{=} e^6(h + O(h^2))$$
$$= e^6 h + O(h^2)$$

2) $$f(2, 2+k) - f(2,2) = e^{2+2(2+k)} - e^{2+4}$$
$$= e^6(e^{2k} - 1)$$
$$= e^6 \cdot 2k + O(k^2)$$

(Kolla att $f(2+h, 2+k) - f(2,2)$ uppfyller liknande likhet)

$\Rightarrow f$ är differentierbar.

Vi ser också att

$$\begin{cases} \dfrac{\partial f}{\partial x}(2,2) = e^6 \\[2ex] \dfrac{\partial f}{\partial y}(2,2) = 2\cdot e^6 \end{cases} \qquad (f(x,y) = e^{x+2y})$$

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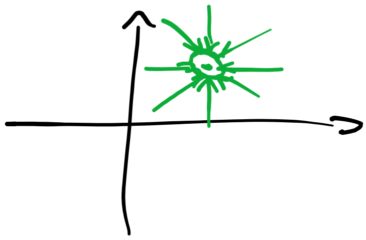
(PAUS) $f(2+h, 2+k) = e^{(2+h)+2(2+h)}$

$= e^{6+h+2k} = e^6 e^{h+2k}$



$f(x,y) = e^x \cdot e^{2y}$

$\sin(x + y^2)$

2.9: Effekt $P(U,R) = \dfrac{U^2}{R}$ och vi antar att $U = 220$ V, $R = 9\ \Omega$. Uppskatta ökningen av effekt, givet att spänningen ökar med 5V och resistansen med 0.3 Ω.

$$\Delta P \approx \frac{\partial P}{\partial U}\Delta U + \frac{\partial P}{\partial R}\Delta R$$

$\Delta U = 5$, $\Delta R = 0.3$, $U = 220$, $R = 9$.

$$\frac{\partial P}{\partial U}(U,R) = \frac{2U}{R} = \frac{440}{9}$$

$$\frac{\partial P}{\partial R}(U,R) = -\frac{U^2}{R^2} = -\left(\frac{220}{9}\right)^2$$

$$\Longrightarrow \Delta P \approx \frac{440}{9}\cdot 5 - \left(\frac{220}{9}\right)^2 \cdot 0.3 \approx 65\ \text{W}.$$

Extra: $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$,

$$f(x,y) = \begin{cases} \dfrac{x^3 y^4}{x^4 + y^{16}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Visa

i) att f är väldefinierad

ii) Visa att alla riktningsderivator av f i origo är noll.

iii) Visa att f ej är kontinuerlig.

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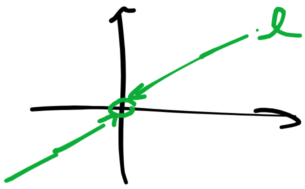
i) Enda (möjliga) problem är nollställen till x^4+y^{16}.

Men, $x^4+y^{16}=0 \Rightarrow (x,y)=(0,0)$.

$\rightsquigarrow$ Inga problem!

ii) Riktningsderivator av f är på formen

$$\frac{\partial f}{\partial \gamma_k} = \frac{d(f\circ\gamma_k)}{dt}, \quad \text{där } \gamma_k : \mathbb{R} \to \mathbb{R}^2, \; t \mapsto (t, kt)$$



$$= \frac{d}{dt}\left(\frac{t^3\cdot k^4t^4}{t^4+k^{16}t^{16}}\right)$$

$$= \frac{7\cdot k^4t^6(t^4+k^{16}t^{16}) - k^4t^7(4t^3+16k^{16}t^{15})}{(t^4+k^{16}t^{16})^2}$$

$$= \frac{t^{10}}{t^8}\cdot\frac{7k^4+7\cdot k^{20}\cdot t^{12}-4k^4-16k^{20}t^{12}}{(1+k^{16}t^{12})^2}$$

$$\sim t^2\,\frac{3k^4}{1^2} = 3k^4t^2 \xrightarrow[t\to 0]{} 0$$

y-axeln: $f(0,y)=0 \Rightarrow \frac{\partial f}{\partial y}(0,0)=0$.

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iii) Parametrisera som i (ii):

$$(f\circ\gamma_k)(t) = f(t,kt) = \frac{k^4t^7}{\ldots}$$

$$(t^4+k^{16}t^{16})$$

$$= \frac{t^7}{t^4} \cdot \frac{k^4}{1+k^{16}t^{12}} \sim k^4 t^3 \xrightarrow[t\to 0]{} 0$$

$$f(0,y) = 0$$

Räcker inte att kolla över linjer!

<u>Idé:</u> $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^2$
$t \longmapsto (t, t^{1/4})$,

$$\Rightarrow (f\circ\gamma)(t) = \frac{t^3(t^{1/4})^4}{t^4+(t^{1/4})^{16}} = \frac{t^4}{t^4+t^4}$$

$$= \frac{1}{2} \not\xrightarrow[t\to 0]{} 0$$

$f\circ\gamma$ är <u>inte</u> kontinuerlig

$$\Downarrow$$

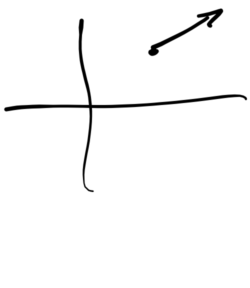
<u>f ej kont.</u> <u>eller</u> ~~γ ej kont.~~

$$\nabla f(a) = \left(\frac{\partial f}{\partial x_1}(a), \dots, \frac{\partial f}{\partial x_n}(a)\right)$$

$\gamma_k : \mathbb{R} \longrightarrow \mathbb{R}^2$
$t \longmapsto (t, kt)$

$$\frac{\partial f}{\partial \gamma_k}(a) = \frac{d(f\circ\gamma_k)}{dt} = \frac{\partial f}{\partial x}(a)\frac{dx}{dt} + \frac{\partial f}{\partial y}(a)\frac{dy}{dt}$$

$$= \frac{\partial f}{\partial x}(a) + k\frac{\partial f}{\partial y}(a)$$



$$= \langle \nabla f(a), (1,k) \rangle \qquad v\cdot u$$

$$\left(\frac{\partial f}{\partial x}(a), \frac{\partial f}{\partial y}(a)\right)$$