MVE035/600 Exercise session 2.1.

Wednesday, 27 January 2021 07:59

2.34: $f(x,y) = xy\sin x$, $(x,y) = (\frac{\pi}{2}, 1)$

a) I vilken riktning växer f som snabbast?

b) Bestäm riktningsderivatan av f i riktningen $(3,4)$

c) Bestäm tangentplanet till ytan $z = f(x,y)$ i $(\frac{\pi}{2}, 1)$.

Lösn. a) Av def. är

$$v \in S^1, \quad \frac{\partial f}{\partial v}(x,y) = \langle \nabla f(x,y), v \rangle$$

$$= \|\nabla f(x,y)\| \cos(\arg(\nabla f(x,y), v))$$

$$\leq \|\nabla f(x,y)\|$$

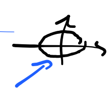
med likhet när $v = \dfrac{\nabla f(x,y)}{\|\nabla f(x,y)\|}$.

$\Rightarrow$ "Brantaste riktning" är

$$\nabla f(\tfrac{\pi}{2}, 1) = \begin{pmatrix} y\sin x + xy\cos x \\ x\sin x \end{pmatrix}\Bigg|_{(\frac{\pi}{2},1)}$$

$$= \begin{pmatrix} 1 \\ \frac{\pi}{2} \end{pmatrix}$$

$$\Rightarrow \frac{\nabla f(\frac{\pi}{2},1)}{\|\nabla f(\frac{\pi}{2},1)\|} = \frac{1}{\sqrt{1+\frac{\pi^2}{4}}} \begin{pmatrix} 1 \\ \frac{\pi}{2} \end{pmatrix}.$$



b) Låt $v = \dfrac{(-3,4)}{\|(-3,4)\|} = \dfrac{1}{5}\begin{pmatrix} -3 \\ 4 \end{pmatrix} \in S^1$

$$\Rightarrow \frac{\partial f}{\partial v}(\tfrac{\pi}{2}, 1) = \langle \nabla f(\tfrac{\pi}{2}, 1), v \rangle$$

$$= \left\langle \begin{pmatrix} 1 \\ \pi/2 \end{pmatrix}, \frac{1}{5}\begin{pmatrix} -3 \\ 4 \end{pmatrix} \right\rangle$$

$$= \frac{1}{5}(-3 + 2\pi)$$

$$= \frac{2\pi - 3}{5}.$$

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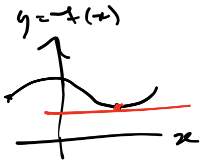
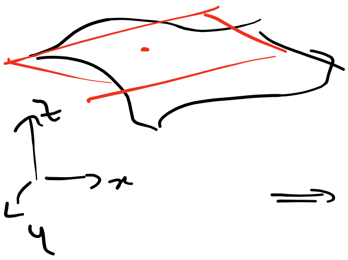
c) Tangentplan till $z = f(x,y)$ i $(\frac{\pi}{2}, 1)$:



$$z - f(\tfrac{\pi}{2},1) = \frac{\partial f}{\partial x}(\tfrac{\pi}{2},1)(x-\tfrac{\pi}{2}) + \frac{\partial f}{\partial y}(\tfrac{\pi}{2},1)(y-1)$$

$$\left(= \left\langle \nabla f(\tfrac{\pi}{2},1), (x-\tfrac{\pi}{2}, y-1)\right\rangle\right)$$



$$\begin{cases} f(\frac{\pi}{2},1) = \frac{\pi}{2}\cdot 1 \cdot \sin\frac{\pi}{2} = \frac{\pi}{2}. \\ \nabla f(\frac{\pi}{2},1) = \begin{pmatrix} 1 \\ \pi/2 \end{pmatrix} \end{cases}$$

$$\Longrightarrow \quad z - \frac{\pi}{2} = x - \frac{\pi}{2} + \frac{\pi}{2}(y-1)$$

$$\Updownarrow$$

$$x + \frac{\pi}{2}y - z = \frac{\pi}{2}$$

Extra: Visa att för $x(s,t) = e^s\cos(t)$, $y(s,t) = e^s \sin(t)$

gäller det att

$(f \in C^2(\mathbb{R}^2))$ $\quad \frac{\partial^2 f}{\partial s^2} + \frac{\partial^2 f}{\partial t^2} = (x^2+y^2)\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right).$

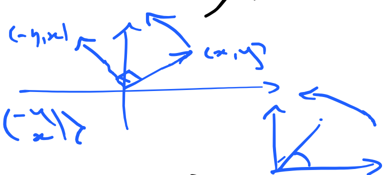
Lös: $\quad \frac{\partial x}{\partial s} = e^s \cos t = x, \quad \frac{\partial y}{\partial s} = y$

$$\frac{\partial x}{\partial t} = -e^s \sin t = -y, \quad \frac{\partial y}{\partial t} = e^s\cos t = x$$

$$= \langle \nabla f, \tbinom{x}{y} \rangle$$

$$\Longrightarrow \begin{cases} \frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}, \\ \frac{\partial f}{\partial t} = -y\frac{\partial f}{\partial x} + x \frac{\partial f}{\partial y} \end{cases}$$

$$= \langle \nabla f, \tbinom{-y}{x} \rangle$$



$(x,y): \mathbb{R}^2 \to \mathbb{R}^2$

$$\Longrightarrow \frac{\partial^2 f}{\partial s^2} = \frac{\partial}{\partial s}\frac{\partial f}{\partial s} = \frac{\partial}{\partial s}\left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right) = x\frac{\partial f}{\partial x} + x\cdot$$

$$\cdot\left(\frac{\partial^2 f}{\partial x^2}\cdot x + y\cdot\frac{\partial^2 f}{\partial y \partial x}\right) + y\frac{\partial f}{\partial y} + \left(\frac{\partial^2 f}{\partial x \partial y}\cdot x + \frac{\partial^2 f}{\partial y^2}\cdot y\right)\cdot y$$

$$= x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + xy\frac{\partial^2 f}{\partial y \partial x} + y\cdot x\frac{\partial^2 f}{\partial x \partial y}$$

$$+ x^2 \frac{\partial^2 f}{\partial x^2} + y^2 \frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t}\left(-y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}\right) =$$

$$= -x\frac{\partial f}{\partial x} - y\left(-y\frac{\partial^2 f}{\partial x^2} + x\frac{\partial^2 f}{\partial y\partial x}\right) - y\frac{\partial f}{\partial y} + x\left(-y\frac{\partial^2 f}{\partial x\partial y} + x\frac{\partial^2 f}{\partial y^2}\right)$$

$$= -x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y} - yx\frac{\partial^2 f}{\partial y\partial x} - xy\frac{\partial^2 f}{\partial x\partial y}$$

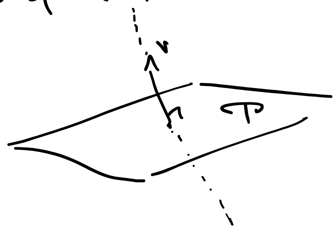
$$+ y^2\frac{\partial^2 f}{\partial x^2} + x^2\frac{\partial^2 f}{\partial y^2}$$

$$\Longrightarrow \frac{\partial^2 f}{\partial s^2} + \frac{\partial^2 f}{\partial t^2} = (x^2+y^2)\underbrace{\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)}_{\Delta f}$$

$$\Delta_{pol.} = \|\cdot\|^2 \Delta_{Euc.}$$

PAUS: Tangentplan: Givet ett plan $P \subset \mathbb{R}^3$ $\exists v \in \mathbb{R}^3$ s.a.

$\mathbb{R}^3 = P \oplus \operatorname{span}\{v\} \leftarrow \{\lambda v : \lambda \in \mathbb{R}\}$



Givet $v \in \mathbb{R}^3$ så är

$\operatorname{span}\{v\}^\perp = \{u \in \mathbb{R}^3 : \langle u, v\rangle = 0\}$

ett plan.

(Linjära) Plan $\underset{\text{ortogonalt komplement}}{\overset{\text{normal}}{\rightleftarrows}}$ linjer (i $\mathbb{R}^3$)

$\{u : \langle u, v\rangle = t\}$

Tangentrum: Yta $f(x,y) = z$, pkt $(p,q) \in \mathbb{R}^2$

$$\Rightarrow \underbrace{-(z - f(p,q)) + \frac{\partial f}{\partial x}(p,q)(x-p) + \frac{\partial f}{\partial y}(p,q)(y-q)}_{=0}$$

$$= \left\langle \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ -1 \end{pmatrix}, \begin{pmatrix} x-p \\ y-q \\ z-f(p,q) \end{pmatrix} \right\rangle$$

2.62. b) $f(x,y) = \sqrt{1+x+y}$, Taylorutveckla upp till grad 2 kring punkten $(1,0)$.

Lösn. $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{1}{2\sqrt{1+x+y}}\Big|_{(1,0)} = \frac{1}{2\sqrt{2}}$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y^2} = -\frac{1}{4(1+x+y)^{3/2}}\Big|_{(1,0)} = -\frac{1}{16\sqrt{2}}$$

$f(1,0) = \sqrt{2}$.

$$\Rightarrow f(1+h,k) = f(1,0) + \frac{\partial f}{\partial x}(1,0)h + \frac{\partial f}{\partial y}(1,0)k$$

$((h,k) \neq (0,0))$
$$+ \frac{\partial^2 f}{\partial x^2}\frac{h^2}{2} + 2\frac{\partial^2 f}{\partial x \partial y}\frac{hk}{2} + \frac{\partial^2 f}{\partial y^2}\cdot\frac{k^2}{2} + O((h+k)^3)$$

$$= \sqrt{2} + \frac{1}{2\sqrt{2}}(h+k) - \frac{1}{32\sqrt{2}}(h+k)^2 + O((h+k)^3)$$

2.67: $f(x,y) = x^3y^2 + 27xy + 27y$,
hitta extrempunkter!

Lösn.
$$\begin{cases} \frac{\partial f}{\partial x} = 3x^2y^2 + 27y \\ \frac{\partial f}{\partial y} = 2x^3y + 27x + 27 \end{cases}$$

$$\frac{\partial^2 f}{\partial x^2} = 6xy^2, \quad \frac{\partial^2 f}{\partial x \partial y} = 6x^2y + 27, \quad \frac{\partial^2 f}{\partial y^2} = 2x^3.$$

Stationära punkter:

$$\nabla f(x,y) = 0 \iff \begin{cases} y(3x^2y + 27) = 0 \\ 2x^3y + 27x + 27 = 0 \end{cases}$$

$y = 0$: $27x + 27 = 0 \Rightarrow x = -1 \Rightarrow (x,y) = (-1,0)$

$y \neq 0$: $3x^2y = -27 \iff y = -\frac{9}{x^2}$ $(x \neq 0)$

$\Rightarrow -18 \cdot x + 27x = -27$

$9x = -27$

$x = -3$ $(\neq 0)$, $y = -1$

$\Rightarrow$ pkterna är $(-1,0)$ och $(-3,-1)$.

Extrempunkter?

Def. $$\mathrm{Hess}(f)(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x,y) & \frac{\partial^2 f}{\partial x \partial y}(x,y) \\ \frac{\partial^2 f}{\partial y \partial x}(x,y) & \frac{\partial^2 f}{\partial y^2}(x,y) \end{pmatrix} =: \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

Sats:
1) $AC - B^2 > 0$, $A > 0 \Rightarrow f$ har lokalt maximum i (x,y)
2) $AC - B^2 > 0$, $A < 0 \Rightarrow f$ —"— minimum i (x,y)
3) $AC - B^2 < 0 \Rightarrow (x,y)$ är en sadelpunkt till f.

$$\mathrm{Hess}(f)(-1,0) = \begin{pmatrix} 0 & 27 \\ 27 & -2 \end{pmatrix},$$
$$\mathrm{Hess}(f)(-3,-1) = \begin{pmatrix} -18 & -54 \\ -54 & -9 \end{pmatrix}$$

$$\Rightarrow \det(\mathrm{Hess}(f)) = \begin{cases} -27^2 < 0 & \text{i } (-1,0) \\ 18 \cdot 9 - 54^2 < 0 & \text{i } (-3,-1) \end{cases}$$

$\Rightarrow$ sadelpunkter!

P-B: $$Q_{f,p}(h,k) = Ah^2 + 2Bhk + Ck^2 = (h\ k) \begin{pmatrix} A & B \\ B & C \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} = \left\langle \begin{pmatrix} h \\ k \end{pmatrix}, \mathrm{Hess}(f)(p) \begin{pmatrix} h \\ k \end{pmatrix} \right\rangle$$