MVE035/600 Exercise session 2.2.

Wednesday, 27 January 2021 15:02

2.34: $f(x,y) = xy\sin(x)$, $(x,y) = (\frac{\pi}{2}, 1)$

a) Bestäm riktningen dit f växer som snabbast.

b) Beräkna riktningsderivatan av f i riktningen $(-3,4)$.

c) Ange ekvation för tangentplanet till ytan $z = f(x,y)$ i $(\frac{\pi}{2}, 1)$.

Lösn. a) $v \in S^1$, $\frac{\partial f}{\partial v}(x,y) = \langle \nabla f(x,y), v \rangle$

$$= \|\nabla f(x,y)\| \cos(\theta),$$

$$\theta = \arg(\nabla f(x,y), v).$$

$\Rightarrow \frac{\partial f}{\partial v}(x,y) \leq \|\nabla f(x,y)\|$ med likhet omm

$$v = \frac{\nabla f(x,y)}{\|\nabla f(x,y)\|}.$$

$$\nabla f(\tfrac{\pi}{2}, 1) = \begin{pmatrix} y\sin(x) + xy\cos(x) \\ x\sin(x) \end{pmatrix}\Bigg|_{(\frac{\pi}{2},1)} = \begin{pmatrix} 1 \\ \frac{\pi}{2} \end{pmatrix}$$

$$\Rightarrow \frac{\nabla f(\frac{\pi}{2},1)}{\|\nabla f(\frac{\pi}{2},1)\|} = \frac{1}{\sqrt{1+\frac{\pi^2}{4}}} \begin{pmatrix} 1 \\ \frac{\pi}{2} \end{pmatrix}$$

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b) $v = \frac{(-3,4)}{\|(-3,4)\|} = \frac{(-3,4)}{5} \in S^1$

$$\Rightarrow \frac{\partial f}{\partial v}(\tfrac{\pi}{2},1) = \langle \nabla f(\tfrac{\pi}{2},1), v \rangle$$

$$= \left\langle \begin{pmatrix} 1 \\ \frac{\pi}{2} \end{pmatrix}, \frac{1}{5}\begin{pmatrix} -3 \\ 4 \end{pmatrix} \right\rangle$$

$$= \frac{2\pi - 3}{5}.$$

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c) "Tangentplanet i $(\frac{\pi}{2}, 1)$" är

$$z - f(\tfrac{\pi}{2}, 1) = \frac{\partial f}{\partial x}(\tfrac{\pi}{2},1)(x - \tfrac{\pi}{2}) + \frac{\partial f}{\partial y}(\tfrac{\pi}{2},1)(y-1)$$

$f(\frac{\pi}{2},1) = \frac{\pi}{2}\cdot 1 \cdot \sin\frac{\pi}{2} = \frac{\pi}{2}$, så ekvationen är

$$z - \frac{\pi}{2} = 1\cdot(x-\frac{\pi}{2}) + \frac{\pi}{2}(y-1)$$

$$\Updownarrow$$

$$x + \frac{\pi}{2}y - z = \frac{\pi}{2}$$

$$\Updownarrow$$

$$\left(\left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} 1 \\ \pi/2 \\ -1 \end{pmatrix} \right\rangle = \frac{\pi}{2} \right)$$

Extra: Låt $x(s,t) = e^s\cos(t)$, $y(s,t) = e^s\sin(t)$, $f \in C^2(\mathbb{R}^2)$. Visa att

$$\frac{\partial^2 f}{\partial s^2} + \frac{\partial^2 f}{\partial t^2} = (x^2+y^2)\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)$$

Lösn. $\frac{\partial x}{\partial s} = e^s\cos(t) = x$, $\frac{\partial y}{\partial s} = y$

$\frac{\partial x}{\partial t} = -e^s\sin(t) = -y$, $\frac{\partial y}{\partial t} = e^s\cos(t) = x$

$$\begin{cases} \frac{\partial f}{\partial s} = \frac{\partial x}{\partial s}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial s}\frac{\partial f}{\partial y} = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial t} = -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} \end{cases}$$

- $$\frac{\partial^2 f}{\partial s^2} = \frac{\partial}{\partial s}\left(x\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial s}\left(y\frac{\partial f}{\partial y}\right)$$
$$= x\frac{\partial f}{\partial x} + x\left(x\frac{\partial^2 f}{\partial x^2} + y\frac{\partial^2 f}{\partial y\partial x}\right) + y\cdot\frac{\partial f}{\partial y} + y\left(x\frac{\partial^2 f}{\partial x\partial y} + y\frac{\partial^2 f}{\partial y^2}\right)$$
$$= x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + 2xy\frac{\partial^2 f}{\partial x\partial y} + x^2\frac{\partial^2 f}{\partial x^2} + y^2\frac{\partial^2 f}{\partial y^2}.$$

- $$\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t}\left(-y\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial t}\left(x\frac{\partial f}{\partial y}\right)$$
$$= -x\frac{\partial f}{\partial x} - y\left(-y\frac{\partial^2 f}{\partial x^2} + x\frac{\partial^2 f}{\partial y\partial x}\right) - y\frac{\partial f}{\partial y}$$

$$+\, x\left(-y\frac{\partial^2 f}{\partial x\partial y}+x\frac{\partial^2 f}{\partial y^2}\right)$$

$$=-x\frac{\partial f}{\partial x}-y\frac{\partial f}{\partial y}-2xy\frac{\partial^2 f}{\partial x\partial y}+y^2\frac{\partial^2 f}{\partial x^2}+x^2\frac{\partial^2 f}{\partial y^2}$$

$$\Rightarrow \frac{\partial^2 f}{\partial s^2}+\frac{\partial^2 f}{\partial t^2}=(x^2+y^2)\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right) \qquad \blacksquare$$

2.62: Taylorutveckla $f(x,y)=\sqrt{1+x+y}$ till grad 2 i punkten $(1,0)$.

Lösn. $\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=\frac{1}{2\sqrt{1+x+y}}\Big|_{(1,0)}=\frac{1}{2\sqrt{2}}$

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y^2}=-\frac{1}{4(1+x+y)^{3/2}}\Big|_{(1,0)}=-\frac{1}{16\sqrt{2}}.$$

För (h,k) nära $(0,0)$ så är

$$f(1+h,k)=f(1,0)+\frac{\partial f}{\partial x}(1,0)h+\frac{\partial f}{\partial y}(1,0)k$$
$$+\frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}(1,0)h^2+2\frac{\partial^2 f}{\partial x\partial y}(1,0)hk+\frac{\partial^2 f}{\partial y}(1,0)k^2\right)$$
$$+\,\mathcal{O}(\|h,k\|^3)$$
$$=\sqrt{2}+\frac{1}{2\sqrt{2}}(h+k)-\frac{1}{32\sqrt{2}}(h+k)^2$$
$$+\,\mathcal{O}((h+k)^3)$$

2.67: Hitta extrempunkter till

$$f(x,y)=x^3y^2+27xy+27y.$$

Lösn.
$$\begin{cases}\frac{\partial f}{\partial x}=3x^2y^2+27y\\[2mm] \frac{\partial f}{\partial y}=2x^3y+27x+27\end{cases}$$

$$\begin{cases}\frac{\partial^2 f}{\partial x^2}=6xy^2\\[2mm] \frac{\partial^2 f}{}=6x^2y+27\end{cases}$$

$$\begin{cases} \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial y^2} = 2x^3 . \end{cases}$$

<u>Stationära punkter:</u> $\nabla f(x,y) = 0$

$$\Longleftrightarrow \begin{cases} y(3x^2y + 27) = 0 \\ 2x^3y + 27x + 27 = 0 \end{cases}$$

1) $y = 0$: $27x + 27 = 0 \implies x = -1$.

2) $y \neq 0$: $3x^2y + 27 = 0 \implies x^2 = -\dfrac{9}{y} \neq 0 \quad (x \neq 0)$

$$\implies \underbrace{2x^3\left(-\frac{9}{x^2}\right)}_{-18x} + 27x + 27 = 0$$

$$\Longleftrightarrow 9x = -27 \Longleftrightarrow x = -3$$

$$\implies y = -\frac{9}{x^2} = -1$$

Stationära punkter är $(-1,0)$ och $(-3,-1)$.

<u>Extrempunkter</u>!?

$$\left\{ \text{\underline{Def}:} \quad \text{Hess}(f)(x,y) = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x^2}(x,y) & \dfrac{\partial^2 f}{\partial x \partial y}(x,y) \\ \dfrac{\partial^2 f}{\partial y \partial x}(x,y) & \dfrac{\partial^2 f}{\partial y^2}(x,y) \end{pmatrix} \right\}$$

$$(f \in C^2(\mathbb{R}^2)) \qquad =: \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

<u>Sats</u>: $f \in C^2(\mathbb{R}^2)$, $(x,y) \in \mathbb{R}^2$ stationär.

1) Om $AC - B^2 > 0$, $A > 0 \implies (x,y)$ lokalt minimum till f

2) Om $AC - B^2 > 0$, $A < 0 \implies (x,y)$ lokalt maximum till f

3) Om $AC - B^2 < 0 \implies (x,y)$ sadelpunkt till f.

$(x,y) = (-1,0)$, $(-3,-1)$,

$$\begin{cases} f_{xx} = 6xy^2 = \begin{cases} 0 \\ -18 \end{cases} \\ f_{xy} = 6x^2y + 27 = \begin{cases} 27 \\ -27 \end{cases} \\ f_{yy} = 2x^3 = \begin{cases} -2 \\ -54 \end{cases} \end{cases}$$

$$\det \mathrm{Hess}(f)(-1,0) = \begin{vmatrix} 0 & 27 \\ 27 & -2 \end{vmatrix} = -27^2 < 0 \quad , \text{ sadel pkt.}$$

$$\det \mathrm{Hess}(f)(-3,-1) = \begin{vmatrix} -18 & -27 \\ -27 & -54 \end{vmatrix} = 18 \cdot 54 - 27^2 > 0$$

$$\Rightarrow (-3,-1) \text{ lokalt maximum.}$$