MVE035/600 Exercise session 3.1.

Wednesday, 3 February 2021 08:01

3.9: d) Bestäm funktionalmatrisen till

$$F(x,y) = (x^2-y^2,\ 2xy) \quad . \quad (= (f(x,y), g(x,y)))$$

Lösn. $$DF(x,y) = \begin{pmatrix} \nabla f(x,y)^t \\ \nabla g(x,y)^t \end{pmatrix}$$

$$= \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}$$

OBS! F är ej globalt injektiv,

$$F(-x,-y) = F(x,y) \quad \forall (x,y) \in \mathbb{R}^2.$$

3.28: $f(x,y,z) = e^{z-1} + zy + x - 2y^3$,

ytan $Y = f^{-1}\{0\}$ och $P = (0,1,1) \in \mathbb{R}^3$.

a) Visa att $P \in Y$ och bestäm tangentplanet till Y i P.

b) Avgör om $\exists g : \mathbb{R}^2 \to \mathbb{R}$ s.a.

$$Y = \{(x,y,z) \in \mathbb{R}^3 : z = g(x,y)\}$$

i en omgivning av P.

Lösning: a)

$$f(P) = f(0,1,1) = e^{1-1} + 1\cdot 1 + 0 - 2\cdot 1^3$$

$$= 2 - 2 = 0.$$

$\Rightarrow P \in Y$.

Tangentplan i P: $\langle \nabla f(P), p - P \rangle = 0$, $p \in \mathbb{R}^3$

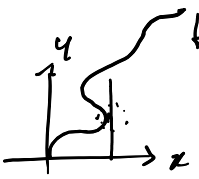
$$\nabla f(P) = \begin{pmatrix} 1 \\ z-6y^2 \\ e^{z-1}+y \end{pmatrix}\Bigg|_{P=(0,1,1)} = \begin{pmatrix} 1 \\ -5 \\ 2 \end{pmatrix}$$

$$\Rightarrow (x,y,z) : \left\langle \begin{pmatrix} 1 \\ -5 \\ 2 \end{pmatrix}, \begin{pmatrix} x-0 \\ y-1 \\ z-1 \end{pmatrix} \right\rangle = 0$$

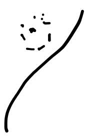
$$\parallel$$

$$x - 5(y-1) + 2(z-1)$$

$$\Leftrightarrow \underline{(x,y,z) : x - 5y + 2z = -3}.$$



b) $\frac{\partial f}{\partial z}(P) = 2 \neq 0$, så z är lokalt en funktion av (x,y) enl. implicita funktionssatsen.



3.33: Visa att ytorna

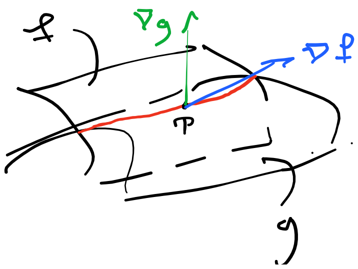
$$\begin{cases} f(x,y,z) = x^2 - y^2 - z^2 + 1 = 0 \\ g(x,y,z) = x^2 + 2y^2 + 3z^2 - 6 = 0 \end{cases}$$

i en omgivning av $P = (1,1,1)$ skär varandra längs en kurva. Bestäm tangentlinjens ekvation i P.

Lösning: $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$(x,y,z) \longmapsto (f(x,y,z), g(x,y,z))$$

och vi är intresserade av $F^{-1}\{0\} \subset \mathbb{R}^3$.



$\nabla f(P) \times \nabla g(P) \neq 0$

$\Rightarrow$ kurvan är implicit definierad i något

$= \quad$ x, y, z - led.

$$\left(\nabla f(P) \times \nabla g(P)\right)\Big|_z = \det D\left(F\big|_{(x,y)}\right)\Big|_{P'=(1,1)}$$

$f = x^2 - y^2 - z^2 + 1$

$g = x^2 + 2y^2 + 3z^2 - 6$

$$= \det \begin{pmatrix} 2x & -2y \\ 2x & 4y \end{pmatrix}\Bigg|_{(1,1)}$$

$$= \det \begin{pmatrix} 2 & -2 \\ 2 & 4 \end{pmatrix}$$

$$= 12 \neq 0$$

$\Rightarrow \exists\, g : \mathbb{R}^2 \to \mathbb{R}$; $F^{-1}\{0\} = \{(x,y,z) : z = g(x,y)\}$,

$g(1,1) = 1$

Tangentlinjen i P: Parametriser $F^{-1}\{0\}$

med $\gamma : \mathbb{R} \to F^{-1}\{0\}$

$t \mapsto (x(t), y(t), t)$, $x, y : \mathbb{R}^2 \to \mathbb{R}$.

<u>OBS</u>: $x(1) = y(1) = 1$.

Tangentlinjen i P er $(1,1,1) + \gamma'(1)\cdot t$, $t \in \mathbb{R}$.

<u>Implicit derivering</u>:

$$\frac{\partial f}{\partial z}(P) = 2x \cdot \frac{dx}{dz} - 2y\frac{dy}{dz} - 2z\Big|_P$$

$$= 2\frac{dx}{dz}(1) - 2\frac{dy}{dz}(1) - 2 = 0 \qquad (1)$$

$$\frac{\partial g}{\partial z}(P) = 2x\frac{dx}{dz} + 4y\frac{dy}{dz} + 6z\Big|_P$$

$$= 2\frac{dx}{dz}(1) + 4\frac{dy}{dz}(1) + 6 = 0 \qquad (2)$$

$$\overset{(1)}{\Rightarrow} \quad \frac{dx}{dz}(1) = \frac{dy}{dz}(1) + 1$$

$$\overset{(2)}{\Rightarrow} \quad 2\frac{dy}{dz}(1) + 2 + 4\frac{dy}{dz}(1) + 6 = 0$$

$$\Rightarrow \quad \frac{dy}{dz}(1) = -\frac{8}{6} = -\frac{4}{3}, \quad \frac{dx}{dz}(1) = -\frac{1}{3}$$

Alltså är $\gamma'(1) = \left(\frac{dx}{dz}(1), \frac{dy}{dz}(1), \frac{dz}{dz}(1)\right)$

$$= \left(-\frac{1}{3}, -\frac{4}{3}, 1\right)$$

$$= -\frac{1}{3}(1, 4, -3).$$

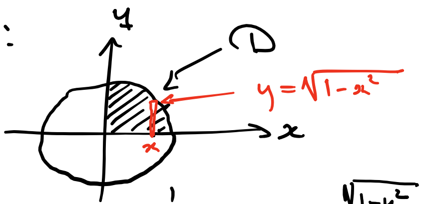
$\rightarrow$ Tangentlinjen $(1,1,1) + t(1,4,-3)$, $t \in \mathbb{R}$.

PAUS: $\left(F^{-1}(\{0\}) = \{ x \in D(F) : F(x) = 0 \} \right)$

6.16: Beräkna $\iint_D \frac{xy}{(1+y^2)^2}\,dx\,dy$,

$$D = \{ (x,y) \in \mathbb{R}^2 : x \geq 0,\ y \geq 0,\ x^2 + y^2 \leq 1 \}$$

Lösn. Bild:



$$\iint_D \frac{xy}{(1+y^2)^2}\,dx\,dy = \int_0^1 \frac{y}{(1+y^2)^2} \int_0^{\sqrt{1-y^2}} x\,dx\,dy$$

$$= \int_0^1 \frac{y}{(1+y^2)^2} \cdot \left(\frac{1-y^2}{2} - 0 \right) dy$$

$$= \{ u = y^2,\ du = 2y\,dy \}$$

$$= \frac{1}{4} \int_0^1 \frac{1-u}{(1+u)^2}\,du$$

$$= \frac{1}{4} \left(\int_0^1 \frac{du}{(1+u)^2} - \int_0^1 \frac{u}{(1+u)^2}\,du \right)$$

$$= \{ v = 1+u,\ dv = du \}$$

$$\frac{1}{4}\Big(\int^2 dv \quad \int^2 v-1 \ \Big)$$

$$= \frac{1}{4}\left(\int_1 \frac{\quad}{v^2} - \int_1 \frac{\quad}{v^2}\, dv \right)$$

$$= \frac{1}{4}\left(2\int_1^2 \frac{dv}{v^2} - \int_1^2 \frac{dv}{v} \right)$$

$$= \frac{1}{4}\left(2\left(-\frac{1}{2}+1\right) - \left(\ln 2 - \cancel{\ln 1}\right) \right)$$

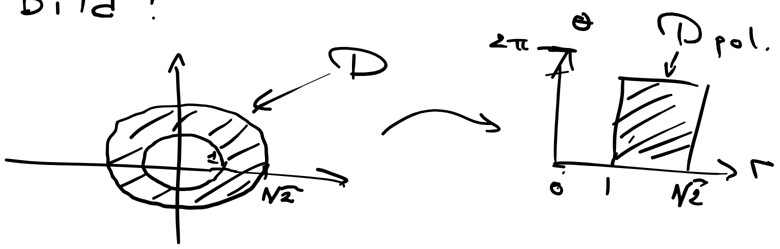
$$= \frac{1}{4}\left(1 - \ln 2\right).$$

6.21: Berechne

$$\iint_D \ln(1+x^2+y^2)\,dx\,dy,$$

$$D = \{ (x,y) \in \mathbb{R}^2 : 1 \le x^2+y^2 \le 2 \}.$$

Lösn. Bild:



$$\Longrightarrow \iint_D \ln(1+x^2+y^2)\,dx\,dy = \left\{ x = r\cos\theta,\ y = r\sin\theta,\ dx\,dy = r\,dr\,d\theta \right\}$$

$$= \int_0^{2\pi}\int_1^{\sqrt{2}} \ln(1+r^2)\, r\,dr\,d\theta$$

$$= 2\pi \int_1^{\sqrt{2}} \ln(1+r^2)\, r\,dr$$

$$= \left\{ t = r^2,\ dt = 2r\,dr \right\}$$

$$= \pi \int_1^2 \ln(1+t)\,dt$$

$$= \pi \int_2^3 \ln(s)\,ds \qquad \left((s\ln(s))' = \ln(s) + s\cdot\frac{1}{s} = \ln(s)+1 \right)$$

$$\left[\quad \right]^3$$

$$= \pi \left[s \ln s - s \right]_2 \qquad = \ln(s) + (s)'$$

$$= \pi \big(3 \ln 3 - 3 \qquad \Rightarrow (s \ln(s) - s)' = \ln(s)$$

$$- (2 \ln 2 - 2) \big)$$

$$= \pi \left(\ln 27 - \ln 4 - 1 \right)$$

$$= \pi \left(\ln \frac{27}{4} - 1 \right) .$$