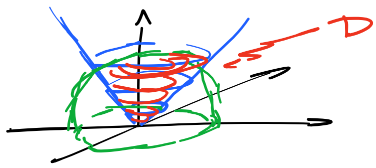
MVE035/600 Exercise session 4.1.

Wednesday, 10 February 2021 07:50

7.4: Beräkna $\displaystyle\iiint_D \frac{z}{1+x^2+y^2}\,dx\,dy\,dz$,

där $D = \{(x,y,z) : x^2+y^2+z^2 \le 1,\ z \ge \sqrt{x^2+y^2}\}$.

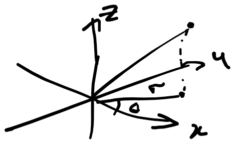
Lösning.



$z = \sqrt{x^2+y^2}$

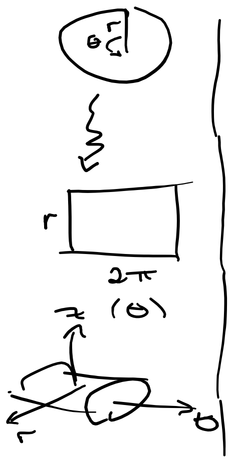
"Cylindriska koordinater":

$$\mathbb{R}^+ \times [0,2\pi) \times \mathbb{R} \longrightarrow \mathbb{R}^3$$
$$(r, \theta, z) \longmapsto (r\cos\theta, r\sin\theta, z)$$



$\rightsquigarrow \quad dx\,dy\,dz = r\,dr\,d\theta\,dz$

$$\iiint_D \frac{z}{1+x^2+y^2}\,dx\,dy\,dz = \iiint_{D_{cyl}} \frac{z}{1+r^2}\,r\,dr\,d\theta\,dz$$



$D_{cyl} = \{(r,\theta,z) : r^2+z^2 \le 1,\ z \ge r\}$

$= \{(r,\theta,z) : r \le z \le \sqrt{1-r^2}\}$

$$= \int_0^{2\pi}\int_0^{1/\sqrt{2}}\int_r^{\sqrt{1-r^2}} \frac{z}{1+r^2}\,r\,dz\,dr\,d\theta$$

$r = \sqrt{1-r^2}$
$r^2 = 1-r^2$
$2r^2 = 1 \Rightarrow r = \frac{1}{\sqrt{2}}$

$$= 2\pi \int_0^{1/\sqrt{2}} \frac{r}{1+r^2}\left[\frac{z^2}{2}\right]_r^{\sqrt{1-r^2}} dr$$

$$= \cancel{2}\pi \int_0^{1/\sqrt{2}} \frac{r}{1+r^2}\cdot\frac{1}{\cancel{2}}\left(1-r^2-r^2\right)dr$$

$$= \pi \int_0^{1/\sqrt{2}} \frac{1-2r^2}{1+r^2}\cdot r\,dr = \frac{\pi}{2}\int_0^{1/2} \frac{1-2u}{1+u}\,du$$

$= \frac{1+u-3u}{1+u}$

$= 1 - 3\cdot\frac{u}{1+u}$

$$= \frac{\pi}{2}\left(\int_0 du - 3\cdot\int_0 \frac{u\,du}{1+u}\right)$$

$$= \left[u\ln(1+u)\right]_0^{1/2} - \int_0^{1/2} \ln(1+u)\,du$$

$$= \frac{1}{2}\ln\left(\frac{3}{2}\right) - \left[(1+u)\left(\ln(1+u)-1\right)\right]_0^{1/2}$$

$$= \frac{1}{2}\ln\left(\frac{3}{2}\right) - \left(\frac{3}{2}\left(\ln\left(\frac{3}{2}\right)-1\right) - 1\left(\ln 1 - 1\right)\right)$$

$$= \frac{1}{2}\ln\left(\frac{3}{2}\right) - \frac{3}{2}\ln\left(\frac{3}{2}\right) + \frac{3}{2} + 1\cdot\cancel{\ln 1} - 1$$

$$= -\ln\left(\frac{3}{2}\right) + \frac{1}{2}.$$

$$= \frac{\pi}{2}\left(\frac{1}{2} - 3\left(\frac{1}{2} - \ln\left(\frac{3}{2}\right)\right)\right) = \frac{\pi}{2}\left(3\ln\left(\frac{3}{2}\right) - 1\right).$$

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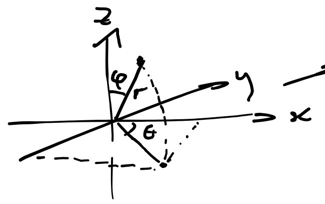
7.15: Beräkna

$$\iiint_{\mathbb{R}^3} \frac{e^{-\|x\|}}{\|x\|}\,d^3x .$$

Lösning:



$(x,y,z) = (r\cos\theta\sin\varphi, r\sin\theta\sin\varphi, r\cos\varphi)$

$r \geq 0$, $\theta \in [0, 2\pi]$, $\varphi \in [0,\pi]$

$\Rightarrow dx\,dy\,dz = r^2 \sin\varphi\,dr\,d\varphi\,d\theta$

$$\iiint_{\mathbb{R}^3} \frac{e^{-\|x\|}}{\|x\|}\,dx = \int_0^\infty \int_0^{2\pi} \int_0^\pi \frac{e^{-r}}{r}\cdot r^2 \sin\varphi\,d\varphi\,d\theta\,dr$$

$$= 2\pi \int_0^\infty r e^{-r}\,dr \cdot \underbrace{\int_0^\pi \sin\varphi\,d\varphi}_{=2}$$

$$= 4\pi\left(\left[-re^{-r}\right]_0^\infty - \int_0^\infty -e^{-r}\,dr\right)$$

$$= 4\pi\left(-\cancel{\lim_{t\to+\infty} te^{-t}} + \left[-e^{-r}\right]_0^\infty\right)$$

$$= 4\pi\left(-\cancel{\lim_{t\to+\infty} e^{-t}} + e^0\right) = 4\pi.$$

8.7: Beräkna volymen av

$$K = \left\{(x,y,z) : \begin{array}{l} 0 \leq z \leq 10 - x^2 - y^2 \\ x + 1 - y^2 \geq 0 \\ x - 1 + y^2 \leq 0 \end{array}\right\}$$

$$= \{(x,y,z) : \begin{array}{l} 0 \le z \le 10-(x^2+y^2) \\ x \ge -(1-y^2) \\ x \le 1-y^2 \end{array} \}$$

$$\rightsquigarrow -(1-y^2) \le 1-y^2$$

$$\mathrm{vol}(K) = \iiint_K dx\,dy\,dz = \int_{-1}^{1} \int_{-(1-y^2)}^{1-y^2} \int_0^{10-(x^2+y^2)} dz\,dx\,dy$$

$$= 4 \int_0^1 \int_0^{1-y^2} \int_0^{10-(x^2+y^2)} dz\,dx\,dy$$

$$= 4 \int_0^1 \int_0^{1-y^2} (10-x^2-y^2)\,dx\,dy$$

$$= 4 \int_0^1 \left((10-y^2)(1-y^2) - \left[\frac{x^3}{3}\right]_0^{1-y^2} \right) dy$$

$$= 4 \int_0^1 \left((10-y^2)(1-y^2) - \frac{1}{3}(1-y^2)^3 \right) dy$$

$$= 4 \int_0^1 \left(10 - 11y^2 + y^4 - \frac{1}{3}(1-3y^2+3y^4-y^6) \right) dy$$

$$= 4 \int_0^1 \left(10 - \frac{1}{3} - 10y^2 + \frac{1}{3}y^6 \right) dy$$

$$= 4 \left(10 - \frac{1}{3} - \frac{10}{3} + \frac{1}{3\cdot 7} \right)$$

$$= 4 \left(\frac{3\cdot 7\cdot 10 - 7\cdot 1 - 7\cdot 10 + 1}{21} \right)$$

$$= 4\,\frac{210-7-70+1}{21} = \frac{4\cdot 134}{21} = \frac{536}{21}.$$

"**Masscentrum**": $K \subset \mathbb{R}^3$ kompakt

$$m_K = \frac{1}{\mathrm{vol}(K)} \left(\int_K x\,dV, \int_K y\,dV, \int_K z\,dV \right).$$

8.31: $K = \{(x,y,z) \in \mathbb{R}^3 : z \le 2-x^2-y^2,\ z \ge y^2\}$

a) Beräkna volymen av K.

b) Beräkna masscentrum av K.

Lösning: a) $y^2 \le z \le 2-y^2-x^2 \implies x^2 \le 2(1-y^2)$

$$\iff |x| \le \sqrt{2(1-y^2)}$$

$$\implies |y| \le 1$$

$$= \mathrm{vol}(K) = \int^{1} \int^{\sqrt{2(1-y^2)}} \int^{2-y^2-x^2}$$

$$\Rightarrow \operatorname{vol}(K) = \int_{-1}^{1}\int_{-\sqrt{2(1-y^2)}}^{\sqrt{2(1-y^2)}}\int_{y^2}^{2-y^2-x^2} dz\,dx\,dy$$

$$= 4\int_0^1\int_0^{\sqrt{2(1-y^2)}}\int_{y^2}^{2-y^2-x^2} dz\,dx\,dy$$

(Jämna funktioner av x och y i integralerna)

$$= 4\int_0^1\int_0^{\sqrt{2(1-y^2)}} \left(2(1-y^2)-x^2\right)dx\,dy$$

$$= 4\int_0^1\left((2(1-y^2))^{3/2} - \frac{1}{3}(2(1-y^2))^{3/2}\right)dy$$

$$= \frac{4\cdot 2^{3/2}\cdot 2}{3}\int_0^1 (1-y^2)^{3/2}dy$$

$$= \left\{ \sin\theta = y,\ \frac{dy}{d\theta} = \cos\theta \right\}$$

$$= \frac{16\sqrt{2}}{3}\int_0^{\pi/2}\cos^4\theta\, d\theta =$$

$$\ldots = \frac{16\sqrt{2}}{3}\cdot\frac{3\pi}{16} = \pi\sqrt{2}.$$

(Verifiera! Hint: Använd att $\cos^2\theta = \frac{1+\cos 2\theta}{2}$ två ggr.)

b) Ur symmetri av K i x-y-planet är x- och y-koordinaterna av m_K $=0$.

$$(m_K)_z = \frac{1}{\operatorname{vol}(K)}\int_K z\,dV,$$

$$\int_K z\,dV = \int_{-1}^{1}\int_{-\sqrt{2(1-y^2)}}^{\sqrt{2(1-y^2)}}\int_{y^2}^{2-x^2-y^2} z\,dz\,dx\,dy$$

$$= 4\int_0^1\int_0^{\sqrt{2(1-y^2)}} \frac{1}{2}\left((2-y^2-x^2)^2 - y^4\right)dx\,dy$$

$$= 2\int_0^1\int_0^{\sqrt{2(1-y^2)}}\left((2-y^2)^2 - 2(2-y^2)x^2 + x^4 - y^4\right)dx\,dy$$

$$= 2\int_0^1\int_0^{\sqrt{2(1-y^2)}}\left(4 - 4y^2 + \cancel{y^4} - 2(2-y^2)x^2 + x^4 - \cancel{y^4}\right)dx\,dy$$

$$= 2\int_0^1\left(2\,(2(1-y^2))^{3/2} - \frac{2}{3}(2-y^2)(2(1-y^2))^{3/2} + \frac{1}{5}(2(1-y^2))^{5/2}\right)dy$$

$$= 4\cdot 2^{3/2}\underbrace{\int_0^{\pi/2}\cos^4\theta\, d\theta}_{=\frac{3\pi}{16}} - \frac{2^{5/2}}{3}\boxed{\int_0^1 (2-y^2)(1-y^2)^{3/2}} \quad (*)$$

$$+ \frac{2^{5/2}}{5}\underbrace{\int_0^{\pi/2}\cos^6\theta\, d\theta}_{=\frac{5\pi}{32}}$$

$= \frac{5\pi}{32}$, samme teknikk som for $\cos^4\theta$, men krever ett extra variabelbyte.

$$(*) = 2\int_0^1 (1-y^2)^{3/2}dy - \int_0^1 y^2(1-y^2)^{3/2}dy$$

$$= 2\underbrace{\int_0^{\pi/2}\cos^4\theta\, d\theta}_{=\frac{3\pi}{16}} - \underbrace{\int_0^{\pi/2}(1-\cos^2\theta)\cos^4\theta\, d\theta}_{=\int_0^{\pi/2}(\cos^4\theta-\cos^6\theta)d\theta \,=\, \frac{3\pi}{16}-\frac{5\pi}{32}}$$

$$= \frac{3\pi}{16} + \frac{5\pi}{32} = \frac{11\pi}{32}$$

Alltså är

$$\int_K z\, dV = \frac{4\cdot 2^{3/2}}{3}\int_0^{\pi/2}\cos^4\theta\, d\theta - \frac{2^{5/2}}{3}\int_0^1 (2-y^2)(1-y^2)^{3/2}dy + \frac{2^{5/2}}{5}\int_0^{\pi/2}\cos^6\theta\, d\theta$$

$$= \frac{8\sqrt{2}}{\cancel{3}}\cdot\frac{\cancel{3}\pi}{16} - \frac{16\sqrt{2}}{3}\left(\frac{3\pi}{16}+\frac{5\pi}{32}\right) + \frac{16\sqrt{2}}{\cancel{5}}\cdot\frac{\cancel{5}\pi}{32}$$

$$= \frac{6\cdot 8\sqrt{2}\pi - 16\sqrt{2}(2\cdot 3\pi + 5\pi) + 3\cdot 16\sqrt{2}\pi}{96}$$

$$= \pi\sqrt{2}\cdot\frac{48 - 16\cdot 11 + 48}{96} = \pi\sqrt{2} = -\pi\sqrt{2}\,\frac{80}{96}$$

$$= -\pi\sqrt{2}\cdot\frac{5}{6}$$

$$\Longrightarrow (m_K)_z = -\frac{5}{6}$$

$$\Longrightarrow m_K = \left(0, 0, -\frac{5}{6}\right). \quad (?)$$

Varför är det negativt ?!?!

I definitionen av K så är $z \geq 0$, så vi borde ha $(m_K)_z \geq 0$. Försök att hitta "felet" i min räkning och deducera att $(m_K)_z = 5/6$.

Hint: Högerhandsregeln.

Svar) a) $\mathrm{vol}(K) = \pi\sqrt{2}$.

b) $m_K = (0, 0, 5/6)$.