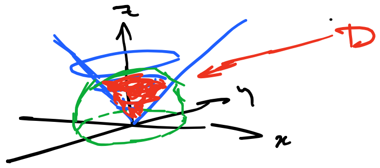
MVE035/600 Exercise session 4.2.

Wednesday, 10 February 2021 15:16

7.4: Beräkna $\iiint_D \frac{z}{1+x^2+y^2}\,dx\,dy\,dz$, där

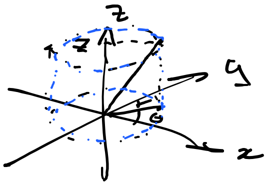
$$D = \{(x,y,z) : x^2+y^2+z^2 \le 1,\ z \ge \sqrt{x^2+y^2}\}.$$

Lösning:



Cylindriska koordinater:

$$\mathbb{R}_{\ge 0} \times [0,2\pi] \times \mathbb{R} \longrightarrow \mathbb{R}^3$$
$$(r,\theta,z) \longmapsto (r\cos\theta, r\sin\theta, z)$$



$$dx\,dy\,dz = r\,dr\,d\theta\,dz$$

$$\iiint_D \frac{z}{1+x^2+y^2}\,dx\,dy\,dz = \iiint_{D_{cyl.}} \frac{z}{1+r^2}\,r\,dr\,d\theta\,dz$$

$$\left(D_{cyl.} = \{(r,\theta,z) : r \le z \le \sqrt{1-r^2}\},\ \Rightarrow r \le \sqrt{1-r^2} \Rightarrow r^2 \le 1-r^2 \Rightarrow 2r^2 \le 1 \Rightarrow r \le \frac{1}{\sqrt{2}} \right)$$

$$= \int_0^{1/\sqrt{2}} \int_0^{2\pi} \int_r^{\sqrt{1-r^2}} \frac{zr}{1+r^2}\,dz\,d\theta\,dr$$

$$= \cancel{2}\pi \int_0^{1/\sqrt{2}} \frac{r}{1+r^2} \left[\frac{z^2}{\cancel{2}}\right]_r^{\sqrt{1-r^2}} dr = \pi \int_0^{1/\sqrt{2}} \frac{r}{1+r^2}(1-2r^2)\,dr$$

$$= \left\{ u = r^2 \Rightarrow \frac{du}{dr} = 2r \Rightarrow r\,dr = \frac{1}{2}du \right\}$$

$$= \frac{\pi}{2} \int_0^{1/2} \frac{1-2u}{1+u}\,du = \frac{\pi}{2}\left(\int_0^{1/2} du - 3\underbrace{\int_0^{1/2} \frac{u}{1+u}\,du}_{(*)}\right)$$

$$\left(= \frac{1+u-3u}{1+u} = 1 - 3\cdot\frac{u}{1+u} \right)$$

$$(*) = \left[u\ln(1+u)\right]_0^{1/2} - \int_0^{1/2} \ln(1+u)\,du = \frac{1}{2}\ln\left(\frac{3}{2}\right)$$

partial int.

$$-\left[(1+u)(\ln(1+u)-1)\right]_0^{1/2}$$

$$= \tfrac{1}{2}\ln\left(\tfrac{3}{2}\right) - \left(\tfrac{3}{2}\left(\ln\tfrac{3}{2}-1\right) - 1(\cancel{\ln 1}-1)\right)$$

$$= -\ln\left(\tfrac{3}{2}\right) + \tfrac{3}{2} - 1 = \tfrac{1}{2} - \ln\left(\tfrac{3}{2}\right).$$

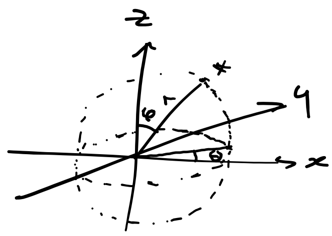
$$= \frac{\pi}{2}\left(\frac{1}{2} - 3\left(\frac{1}{2} - \ln\left(\frac{3}{2}\right)\right)\right) = \frac{\pi}{2}\left(3\ln\left(\frac{3}{2}\right)-1\right).$$

7.15: Beräkna $\displaystyle\iiint_{\mathbb{R}^3} \frac{e^{-\|x\|}}{\|x\|}\,dx$.

Lösning: Sfäriska koordinater:

$$\mathbb{R}_{\geq 0}\times[0,2\pi]\times[0,\pi] \longrightarrow \mathbb{R}^3$$

$$(r,\theta,\varphi) \longmapsto (r\cos\theta\sin\varphi,\ r\sin\theta\sin\varphi,\ r\cos\varphi)$$



$$dx\,dy\,dz = r^2\sin\varphi\,dr\,d\theta\,d\varphi$$

$$\Longrightarrow \iiint_{\mathbb{R}^3}\frac{e^{-\|x\|}}{\|x\|}\,dx = \int_0^\infty\int_0^{2\pi}\int_0^{\pi} \frac{e^{-r}}{r}\,r^2\sin\varphi\,d\varphi\,d\theta\,dr$$

$$= 2\pi\int_0^\infty r e^{-r}\,dr \cdot \underbrace{\int_0^\pi \sin\varphi\,d\varphi}_{\substack{=-\cos(\pi)+\cos(0)\\ =-(-1)+1=2}}$$

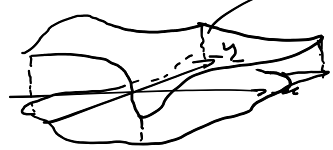
$$= 4\pi\int_0^\infty r e^{-r}\,dr$$

$$\int_0^\infty r e^{-r}\,dr \underset{\text{(partialint.)}}{=} \left[-re^{-r}\right]_0^\infty - \int_0^\infty -e^{-r}\,dr$$

$$= -\lim_{r\to+\infty}\frac{r}{e^r} - \left[e^{-r}\right]_0^\infty = -0-(0-e^0)$$

$$= e^0 = 1$$

$$\Longrightarrow \iiint_{\mathbb{R}^3}\frac{e^{-\|x\|}}{\|x\|}\,dx = 4\pi.$$



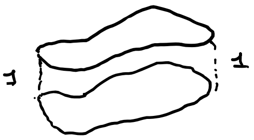
8.7: Beräkna volumen av

$$K = \left\{ (x,y,z) \in \mathbb{R}^3 : \begin{array}{l} 0 \le z \le 10 - x^2 - y^2 \\ x + 1 - y^2 \ge 0 \\ x - 1 + y^2 \le 0 \end{array} \right\}.$$

Lösning: $K = \{ (x,y,z) : 0 \le z \le 10 - x^2 - y^2 , \underbrace{-(1-y^2) \le x \le 1 - y^2}\}$

$$\left(\begin{array}{l} -(1-y^2) \le 1 - y^2 \\ 2y^2 \le 2 \iff y^2 \le 1 \\ \iff -1 \le y \le 1 \end{array} \right)$$

$$\Longrightarrow \mathrm{vol}(K) = \iiint_K dx\,dy\,dz = \int_{-1}^{1} \int_{-(1-y^2)}^{1-y^2} \int_0^{10-x^2-y^2} dz\,dx\,dy$$



$$= \int_{-1}^{1} \int_{-(1-y^2)}^{1-y^2} (10 - x^2 - y^2)\, dx\,dy$$

$$= 4 \int_0^1 \int_0^{1-y^2} (10 - x^2 - y^2)\, dx\,dy$$

$$= 4 \int_0^1 \left((10-y^2)(1-y^2) - \tfrac{1}{3}(1-y^2)^3 \right) dy$$

$$= 4 \int_0^1 \left(10 - 11y^2 + y^4 - \tfrac{1}{3}(1 - 3y^2 + 3y^4 - y^6) \right) dy$$

$$= 4 \int_0^1 \left(10 - \tfrac{1}{3} - 10y^2 + \tfrac{1}{3} y^6 \right) dy$$

$$= 4 \left(10 - \tfrac{1}{3} - \tfrac{10}{3} + \tfrac{1}{3 \cdot 7} \right) = 4 \, \frac{3 \cdot 7 \cdot 10 - 7 - 70 + 1}{21}$$

$$= 4 \, \frac{210 - 70 - 6}{21} = \frac{4 \cdot 134}{21} = \underline{\underline{\frac{536}{21}}}.$$

Masscentrum: $K \subset \mathbb{R}^3$ kompakt

$$m_K := \frac{1}{\mathrm{vol}(K)} \left(\int_K x\,dV, \int_K y\,dV, \int_K z\,dV \right)$$

$$\text{"="} \frac{1}{\mathrm{vol}(K)} \int_K \mathbf{x}\,dV(\mathbf{x})$$

Andra namn är "barycenter av K" eller "väntevärdet av V".

8.31 : $K = \{ (x,y,z) \in \mathbb{R}^3 : z \le 2 - x^2 - y^2 , z \ge y^2 \}$

a) Beräkna volymen av K.

b) Beräkna masscentrum av K.

Lösning: a) $y^2 \le z \le 2 - y^2 - x^2 \Rightarrow x^2 \le 2(1-y^2)$

$\Leftrightarrow |x| \le \sqrt{2(1-y^2)}$

$\Rightarrow |y| \le 1$

$$\Rightarrow \mathrm{vol}(K) = \int_{-1}^{1} \int_{-\sqrt{2(1-y^2)}}^{\sqrt{2(1-y^2)}} \int_{y^2}^{2-y^2-x^2} dz\,dx\,dy$$

$$= 4 \int_0^1 \int_0^{\sqrt{2(1-y^2)}} \int_{y^2}^{2-y^2-x^2} dz\,dx\,dy$$

(Jämna funktioner av x och y i integralerna)

$$= 4 \int_0^1 \int_0^{\sqrt{2(1-y^2)}} (2(1-y^2) - x^2)\,dx\,dy$$

$$= 4 \int_0^1 \left((2(1-y^2))^{3/2} - \frac{1}{3}(2(1-y^2))^{3/2} \right) dy$$

$$= \frac{4 \cdot 2^{3/2} \cdot 2}{3} \int_0^1 (1-y^2)^{3/2}\,dy$$

$$= \left\{ \sin\theta = y,\ \frac{dy}{d\theta} = \cos\theta \right\}$$

$$= \frac{16\sqrt{2}}{3} \int_0^{\pi/2} \cos^4\theta\,d\theta =$$

$$\ldots = \frac{16\sqrt{2}}{3} \cdot \frac{3\pi}{16} = \pi\sqrt{2}.$$

Verifiera!
Hint: Använd att $\cos^2\theta = \frac{1+\cos 2\theta}{2}$
tva ggr.

b) Ur symmetri av K i x-y-planet är x- och y-koordinaterna av $m_K = 0$.

$$(m_K)_z = \frac{1}{\mathrm{vol}(K)} \int_K z\,dV,$$

$$\int_K z\,dV = \int_{-1}^{1} \int_{-\sqrt{2(1-y^2)}}^{\sqrt{2(1-y^2)}} \int_{y^2}^{2-x^2-y^2} z\,dz\,dx\,dy$$

$$= 4 \int_0^1 \int_0^{\sqrt{2(1-y^2)}} \frac{1}{2}\left((2-y^2-x^2)^2 - y^4 \right) dx\,dy$$

$$= 2 \int \int^{\sqrt{2(1-y^2)}} \left((2-y^2)^2 - 2(2-y^2)x^2 + x^4 - y^4 \right) dx\,dy$$

$$= 2\int_0^1\int_0^{\sqrt{2(1-y^2)}} \left(4 - 4y^2 + \cancel{y^4} - 2(2-y^2)x^2 + x^4 - \cancel{y^4}\right)dx\,dy$$

$$= 2\int_0^1 \left(2\,(2(1-y^2))^{3/2} - \frac{2}{3}(2-y^2)(2(1-y^2))^{3/2} + \frac{1}{5}(2(1-y^2))^{5/2}\right) \cdot dy$$

$$= 4\cdot 2^{3/2}\underbrace{\int_0^{\pi/2}\cos^4\theta\,d\theta}_{=\frac{3\pi}{16}} - \frac{2^{5/2}}{3}\boxed{\int_0^1 (2-y^2)(1-y^2)^{3/2}}\ (*) + \frac{2^{5/2}}{5}\underbrace{\int_0^{\pi/2}\cos^6\theta\,d\theta}_{=\frac{5\pi}{32}}$$

$= \frac{5\pi}{32}$, samma teknik som för $\cos^4\theta$, men kräver ett extra variabelbyte.

$$(*) = 2\int_0^1 (1-y^2)^{3/2}dy - \int_0^1 y^2(1-y^2)^{3/2}dy$$

$$= 2\underbrace{\int_0^{\pi/2}\cos^4\theta\,d\theta}_{=\frac{3\pi}{16}} - \underbrace{\int_0^{\pi/2}(1-\cos^2\theta)\cos^4\theta\,d\theta}_{=\int_0^{\pi/2}(\cos^4\theta - \cos^6\theta)\,d\theta \;=\; \frac{3\pi}{16} - \frac{5\pi}{32}}$$

$$= \frac{3\pi}{16} + \frac{5\pi}{32} = \frac{11\pi}{32}$$

Alltså är

$$\int_K z\,dV = \frac{4\cdot 2^{3/2}}{3}\int_0^{\pi/2}\cos^4\theta\,d\theta - \frac{2^{5/2}}{3}\int_0^1 (2-y^2)(1-y^2)^{3/2}dy + \frac{2^{5/2}}{5}\int_0^{\pi/2}\cos^6\theta\,d\theta$$

$$= \frac{8\sqrt{2}}{\cancel{3}}\cdot\frac{\cancel{3}\pi}{16} - \frac{16\sqrt{2}}{3}\left(\frac{3\pi}{16} + \frac{5\pi}{32}\right) + \frac{16\sqrt{2}}{\cancel{5}}\cdot\frac{\cancel{5}\pi}{32}$$

$$= \frac{6\cdot 8\sqrt{2}\pi - 16\sqrt{2}(2\cdot 3\pi + 5\pi) + 3\cdot 16\sqrt{2}\pi}{96}$$

$$= \pi\sqrt{2}\cdot\frac{48 - 16\cdot 11 + 48}{96} = \pi\sqrt{2} = -\pi\sqrt{2}\,\frac{80}{96}$$

$$= -\pi\sqrt{2}\cdot\frac{5}{.}$$

$$\Longrightarrow (m_K)_z = -\frac{5}{6}$$

$$\Longrightarrow m_K = (0,0,-\tfrac{5}{6}). \quad (?)$$

Varför är det negativt?!?!

I definitionen av K så är $z \geq 0$, så vi borde ha $(m_K)_z \geq 0$. Försök att hitta "felet" i min räkning och deducera att $(m_K)_z = 5/6$.

Hint: Högerhandsregeln.

Svar) a) $\mathrm{vol}(K) = \pi\sqrt{2}$.

b) $m_K = (0,0,5/6)$.