MVE035/600 Exercise session 5.2.

Wednesday, 17 February 2021 15:13

Extra: $r : [-1,1] \longrightarrow \mathbb{R}^2$

$t \longmapsto (t^2-2t, t^2+2t)$.

a) Rita $r[-1,1]$.

b) Beräkna hastighet + fart i $t=\frac{1}{3}$.

c) Beräkna längden $\ell(r)$ av $r[-1,1]$.

Lösn. a) $x(t) = t^2-2t$, $y(t) = t^2+2t$

"Tips": $\Rightarrow$ $x(t) = y(-t)$.

b) Hastighet: $r'(\frac{1}{3}) = (2t-2, 2t+2)\Big|_{t=\frac{1}{3}}$

$= (-\frac{4}{3}, \frac{8}{3})$

Fart: $\|r'(\frac{1}{3})\| = \frac{1}{3}\sqrt{16+64} = \frac{\sqrt{80}}{3} = \frac{4\sqrt{5}}{3}$.

c) $\ell(r) = \int_{-1}^{1} \|r'(t)\| dt = 2\int_{-1}^{1} \sqrt{(t-1)^2+(t+1)^2}\, dt$

$= 4\int_0^1 \sqrt{2t^2+2}\, dt = 4\sqrt{2}\int_0^1 \sqrt{t^2+1}\, dt$

Partialintegrering:

$\int_0^1 \sqrt{t^2+1}\, dt = [t\sqrt{t^2+1}]_0^1 - \int_0^1 t \cdot \frac{t}{\sqrt{t^2+1}}\, dt$

$= \sqrt{2} - \int_0^1 (\frac{t^2+1}{\sqrt{t^2+1}} - \frac{1}{\sqrt{t^2+1}})dt = \sqrt{2} + \int_0^1 \frac{dt}{\sqrt{t^2+1}} - \int_0^1 \sqrt{t^2+1}\, dt$

$\int_0^1 \frac{dt}{\sqrt{t^2+1}} = \{t = \tan\theta,\ dt = \frac{d\theta}{\cos^2\theta}\} = \int_0^{\pi/4} \frac{\cos\theta}{\cos^2\theta} d\theta$

$(= (1+\tan^2\theta)d\theta)$

$1+\tan^2\theta = \frac{1}{\cos^2\theta}$

$$= \int_0 \frac{d\theta}{\cos\theta} = \{ u = \sin\theta,\ du = \cos\theta\, d\theta \}$$

$$= \int_0^{1/\sqrt{2}} \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{\sqrt{1-u^2}} = \int_0^{1/\sqrt{2}} \frac{du}{1-u^2} =$$

$$= \int_0^{1/\sqrt{2}} \frac{1}{2}\left(\frac{1}{1+u} + \frac{1}{1-u}\right) du = \frac{1}{2}\Big[\underbrace{\ln(1+u) - \ln(1-u)}_{= \ln\left(\frac{1+u}{1-u}\right)}\Big]_0^{1/\sqrt{2}}$$

$$= \frac{1}{2}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

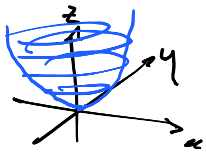
$$\Rightarrow \int_0^1 \sqrt{t^2+1}\, dt = \frac{1}{2}\left(\sqrt{2} + \frac{1}{2}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\right)$$

$$\Rightarrow \ell(r) = 4\sqrt{2}\int_0^1 \sqrt{t^2+1}\, dt = 2\sqrt{2}\left(\sqrt{2} + \frac{1}{2}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\right)$$

$$= 4 + \sqrt{2}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right).$$

3.7: $f : [0,2]\times[0,2\pi] \longrightarrow \mathbb{R}^3$

$$(s,t) \longmapsto (s\cos t,\ s\sin t,\ s^2)$$



$P = (0,1,1) \in f([0,2]\times[0,2\pi]) =: Y$

Beräkna normalen till Y i P.

Lösn. $N_Y(P) = \frac{\partial f}{\partial s}(f^{-1}(P)) \times \frac{\partial f}{\partial t}(f^{-1}(P))$

$P = (0,1,1) = (1\cdot\cos\frac{\pi}{2}, 1\cdot\sin\frac{\pi}{2}, 1^2) = f(1,\frac{\pi}{2}) \Rightarrow f^{-1}(P) = (1,\frac{\pi}{2}).$

$$\Rightarrow N_Y(P) = \begin{pmatrix} \cos t \\ \sin t \\ 2s \end{pmatrix} \times \begin{pmatrix} -s\cdot\sin t \\ s\cdot\cos t \\ 0 \end{pmatrix}\Bigg|_{(s,t)=(1,\frac{\pi}{2})}$$

$$= \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \perp Y.$$

PAUS till 16:14

8.16: $Y = \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2 = R^2,\ z \geq h\}$

där $R \geq 0$ och $0 \leq h \leq R$.

a) Parametrisera Y m.h.a. planet.

b) Beräkna Y:s normalvektorer.

c) Beräkna arean av Y.

<u>Lösn.</u> a) $f : [0,2\pi]\times[h,R] \longrightarrow Y$

$$(\theta, z) \longmapsto \left(\sqrt{R^2-z^2}\cos\theta, \sqrt{R^2-z^2}\sin\theta, z\right)$$

b)
$$N_f(\theta,z) = \frac{\partial f}{\partial \theta}(\theta,z)\times\frac{\partial f}{\partial z}(\theta,z) = \begin{pmatrix} -\sqrt{R^2-z^2}\sin\theta \\ \sqrt{R^2-z^2}\cos\theta \\ 0 \end{pmatrix} \times \begin{pmatrix} \frac{-z}{\sqrt{R^2-z^2}}\cos\theta \\ \frac{-z}{\sqrt{R^2-z^2}}\sin\theta \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} \times \begin{pmatrix} -z\cos\theta \\ -z\sin\theta \\ \sqrt{R^2-z^2} \end{pmatrix} = \begin{pmatrix} \sqrt{R^2-z^2}\cos\theta \\ \sqrt{R^2-z^2}\sin\theta \\ z \end{pmatrix}$$

$$= f(\theta,z) \quad !!$$

c)
$$\mathrm{Area}(Y) = \int_h^R\int_0^{2\pi} \underbrace{\| N_f(\theta,z)\|}_{=R} \, d\theta\, dz = R\int_h^R\int_0^{2\pi} d\theta\, dz$$

$$= 2\pi R(R-h).$$

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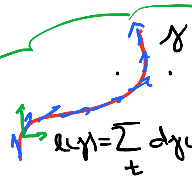
<u>9.4</u>: Låt $\gamma : [1,2] \longrightarrow \mathbb{R}^2$ och beräkna

$$t \longmapsto (t,t^2)$$

$$\int_\gamma y\ln\frac{x^2}{y}dx - \frac{x}{y}dy.$$



$\ell(\gamma) = \sum_t d\gamma(t)$

$= \sum_t a(t)dx(t) + b(t)dy(t)$

<u>Lösn.</u> $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $F(x,y) = \left(y\ln\frac{x^2}{y}, -\frac{x}{y}\right)$

$$\Rightarrow \int_\gamma y\ln\frac{x^2}{y}dx - \frac{x}{y}dy = \int_\gamma \langle F, d\gamma\rangle \overset{\text{def}}{=} \boxed{\int_1^2 \langle F(\gamma(t)), \gamma'(t)\rangle dt}$$

$$= \int_1^2 \langle (\underbrace{\cancel{t^2\ln 1}}_{=0}, -\tfrac{1}{t}), (1,2t)\rangle dt$$

$$= \int_1^2 \left(0\cdot 1 - \frac{2t}{t}\right)dt = \int_1^2 (-2)\,dt = -2$$