MVE035/600 Exercise session 7.1.

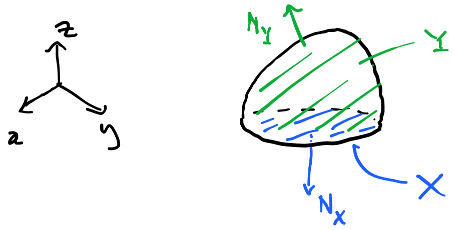
Wednesday, 3 March 2021 07:44

10.11: Låt $u(x,y,z) = (x, y, z+1)$ och beräkna

$$\iint_Y \langle u, N \rangle \, dA,$$

där $Y = \{(x,y,z) : z = 1 - x^2 - y^2, z \geq 0\}$.

Lösn.



K, $\partial K = Y \cup X$.

$K = \{(x,y,z) : 0 \leq z \leq 1 - x^2 - y^2\}$

Gauss:

$$\iint_Y \langle u, N_Y \rangle \, dA = \iiint_K \operatorname{div} u \, dV \quad \leftarrow (1)$$

$$- \iint_X \langle u, N_X \rangle \, dA \quad \leftarrow (2)$$

$$\begin{cases} \operatorname{div} u = 1 + 1 + 1 = 3 \\ N_X = (0, 0, -1) \end{cases} \qquad u = (x, y, z+1)$$

(1) $$\iiint_K \operatorname{div} u \, dV = 3 \cdot \iiint_K dV = 3 \cdot \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{1-x^2-y^2} dz \, dy \, dx$$

$K = \{(x,y,z) : 0 \leq z \leq 1 - x^2 - y^2\}$

$0 \leq 1 - x^2 - y^2$

$y^2 \leq 1 - x^2$

$-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$

$$= 3 \cdot 4 \int_0^1 \int_0^{\sqrt{1-x^2}} (1 - x^2 - y^2) \, dy \, dx$$

$$= 12 \cdot \int_0^1 \left((1-x^2)\sqrt{1-x^2} - \frac{1}{3}(1-x^2)^{3/2} \right) dx$$

$$= 12 \cdot \frac{2}{3} \int_0^1 (1-x^2)^{3/2} \, dx$$

$$= \{ x = \sin\theta, \; dx = \cos\theta \, d\theta \}$$

$$= 8 \int_0^{\pi/2} \cos^3\theta \cdot \cos\theta \, d\theta$$

$$= 8 \int_0^{\pi/2} \cos^4\theta \, d\theta = \ldots = 8 \cdot \frac{3\pi}{16}$$

$$= \frac{3\pi}{2}$$

(2) $$\iint_X \langle u, N_X \rangle \, dA = \iint \left\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\rangle dA(x,y,z)$$

$$= \Big/ X = \{(x,y,z) : x^2+y^2 \le 1,\ z=0\} \Big/$$

$$= \iint_{x^2+y^2\le 1} (-1)\,dx\,dy = -\text{Area}(X)$$

$$= -\pi\,.$$

$$\Rightarrow \iint_Y \langle u, N_Y\rangle\, dA \underset{\text{Gauss}}{=} (1) - (2) = \frac{3\pi}{2} - (-\pi) = \frac{5\pi}{2}.$$

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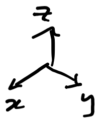
<u>10.23</u>: $K = \{(x,y,z) : 2 \le x^2+y^2+z^2 \le 3\}$,

$$u(x,y,z) = \frac{1}{x^2+y^2+z^2}\begin{pmatrix} x \\ y \\ z\end{pmatrix} \text{ "}=\text{" } \text{"}\frac{\mathbf{r}}{r^2}\text{"}.$$

Beräkna flödet av u genom ∂K.

<u>Lösn.</u>



$K =$

$\partial K =$ sfär av radie $\sqrt{3}$

$\cup$ sfär av radie $\sqrt{2}$

$=: S_3 \cup S_2$

$$\Rightarrow \begin{cases} N_3 := N_{S_3} = \hat{\mathbf{r}} = \dfrac{1}{\sqrt{x^2+y^2+z^2}}\begin{pmatrix} x \\ y \\ z\end{pmatrix} = \dfrac{\mathbf{r}}{r} \\[2ex] N_2 := N_{S_2} = -\dfrac{1}{\sqrt{x^2+y^2+z^2}}\begin{pmatrix} x \\ y \\ z\end{pmatrix} = -\dfrac{\mathbf{r}}{r} \end{cases}$$

$$\Rightarrow \iint_{\partial K} \langle u, N_K\rangle\, dA = \iint_{S_3} \langle u, N_{S_3}\rangle\, dA + \iint_{S_2} \langle u, N_{S_2}\rangle\, dA$$

$$\underset{\text{(Sfäriska koord.)}}{=} \int_0^{2\pi}\int_0^{\pi} \left\langle \frac{\mathbf{r}}{r^2}, \frac{\mathbf{r}}{r}\right\rangle r^2 \sin\varphi\, d\varphi\, d\theta \,\Big|_{r=\sqrt{3}}$$

$$+ \int_0^{2\pi}\int_0^{\pi} \left\langle \frac{\mathbf{r}}{r^2}, \frac{-\mathbf{r}}{r}\right\rangle r^2 \sin\varphi\, d\varphi\, d\theta \,\Big|_{r=\sqrt{2}}$$

$$= \left\{ \left\langle \frac{\mathbf{r}}{r}, \frac{\mathbf{r}}{r}\right\rangle = \left\| \frac{\mathbf{r}}{r}\right\|^2 = 1 \right\}$$

$$= 2\pi\left(\sqrt{3}\underbrace{\int_0^{\pi} \sin\varphi\, d\varphi}_{=2} - \sqrt{2}\int_0^{\pi}\sin\varphi\, d\varphi\right)$$

$$= 4\pi(\sqrt{3}-\sqrt{2})$$

PAUS till 08:58!

10.35: $u(x,y,z) = (x-3y+z^2,\ 2x-y^2+z,\ x^2+y^2-2z^2)$

Beräkna

a) $\operatorname{div} u = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z} = \operatorname{tr}(J u)$

$= 1 - 2y - 4z$

b) $\operatorname{rot} u = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \times \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z} \\ \frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x} \\ \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \end{pmatrix}$

$= \begin{pmatrix} 2y-1 \\ 2z-2x \\ 2-(-3) \end{pmatrix} = \begin{pmatrix} 2y-1 \\ 2(z-x) \\ 5 \end{pmatrix}$

c) ~~grad u~~ = ??? TRICK QUESTION!!!
(inte väldefinierad)

d) $\operatorname{grad} \operatorname{div} u = \operatorname{grad}(1-2y-4z)$

$= \begin{pmatrix} 0 \\ -2 \\ -4 \end{pmatrix} = -2\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$

e) $\operatorname{rot} \operatorname{rot} u = \begin{pmatrix} 0-2 \\ 0-0 \\ -2-2 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -4 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

10.54: M.h.a. Stokes sats, beräkna

$$\int_\gamma (3x+z\cos(yz))\,dy + (y-2x+y\cos(yz))\,dz$$

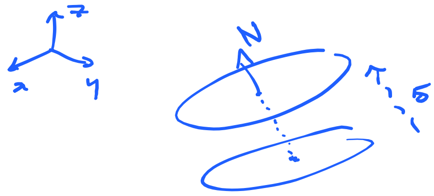
där $\gamma = \partial K$, $K \subset \{(x,y,z) : 2x+y+2z=5\}$ kompakt med $\operatorname{Area}(K) = 3$.

Lösn. $F(x,y,z) := (0,\ 3x+z\cos(yz),\ y-2x+y\cos(yz))$.

$$\underline{\text{Stokes}}: \quad \int_{\partial K} \langle F, d\sigma \rangle = \iint_K \langle \operatorname{rot} F, N_K \rangle \, dA$$

$$\text{Lin. Alg}: \quad N_K = \frac{\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}}{\left\| \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right\|} = \frac{1}{3} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$$\left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right\rangle = 5$$



$$\operatorname{rot} F = \begin{pmatrix} \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{pmatrix} = \Big(1 + \cos(yz) - yz\sin(yz) - (\cos(yz) - yz\sin(yz))$$

$$= \begin{pmatrix} 1 + \cos(yz) - yz\sin(yz) - (\cos(yz) - yz\sin(yz)) \\ 0 - (-2) \\ 3 - 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\Longrightarrow \iint_K \langle \operatorname{rot} F, N_K \rangle \, dA = \iint_K \left\langle \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \frac{1}{3} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right\rangle dA$$

$$= \frac{1}{3} (2 + 2 + 3 \cdot 2) \underbrace{\text{Area}(K)}_{=3}$$

$$= \underline{10}.$$