MVE035/600 Exericise session 7.2.

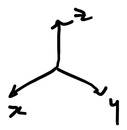
Wednesday, 3 March 2021 09:52

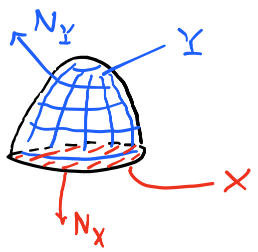
10.11: Låt $u(x,y,z) = (x,y,z+1)$. Beräkna

$$\iint_Y \langle u, N \rangle \, dA$$

där $Y = \{(x,y,z) : z = 1 - x^2 - y^2, z \geq 0\}$.

Lösn.





$$\left\} \begin{aligned} &K = \{(x,y,z) : 0 \leq z \leq 1 - x^2 - y^2\}, \\ &\partial K = X \cup Y. \end{aligned} \right.$$

Gauss: $$\iint_Y \langle u, N_Y \rangle \, dA = \underbrace{\iiint_K \operatorname{div} u \, dV}_{(1)} - \underbrace{\iint_X \langle u, N_X \rangle \, dA}_{(2)}$$

$$\begin{cases} \operatorname{div} u = 1 + 1 + 1 = 3 \\ N_X = (0,0,-1). \end{cases}$$

(1) $$\iiint_K \operatorname{div} u \, dV = 3 \cdot \iiint_K dV = 3 \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{1-x^2-y^2} dz \, dy \, dx$$

$$\left[\begin{aligned} &0 \leq z \leq 1 - x^2 - y^2 \\ &\Downarrow \\ &0 \leq 1 - x^2 - y^2 \\ &\Updownarrow \\ &-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}, \\ &-1 \leq x \leq 1 \end{aligned} \right]$$

$$= 3 \cdot 4 \int_0^1 \int_0^{\sqrt{1-x^2}} (1 - x^2 - y^2) \, dy \, dx = 12 \int_0^1 \left((1-x^2)\sqrt{1-x^2} - \frac{1}{3}(1-x^2)^{3/2} \right) dx$$

$$= 12 \cdot \frac{2}{3} \int_0^1 (1-x^2)^{3/2} \, dx = \{ x = \sin\theta, \, dx = \cos\theta \, d\theta \}$$

$$= 8 \cdot \int_0^{\pi/2} \cos^3\theta \cdot \cos\theta \, d\theta = 8 \cdot \underbrace{\int_0^{\pi/2} \cos^4\theta \, d\theta}_{= \frac{3\pi}{16}} = \ldots = \frac{3\pi}{2}.$$

(2) $$\iint \langle u, N_X \rangle \, dA = \iint \left\langle \begin{pmatrix} x \\ y \\ \ldots \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \ldots \end{pmatrix} \right\rangle dA$$

$$\cdots \iint\limits_{X} \cdots \iint\limits_{x^2+y^2\leq 1} \cdots$$

$$= - \text{Area}\{(x,y) : x^2+y^2 \leq 1\}$$

$$= -\pi \cdot 1^2 = -\pi$$

Alltså,

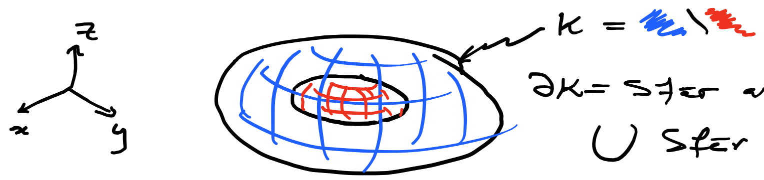
$$\iint\limits_{Y} \langle u, N_Y \rangle dA \underset{\text{Gauss}}{=} (1) - (2) = \frac{3\pi}{2} - (-\pi) = \frac{5\pi}{2}.$$

10.23: $K = \{(x,y,z) : 2 \leq x^2+y^2+z^2 \leq 3\}$,

$$u(x,y,z) = \frac{1}{x^2+y^2+z^2} \cdot (x,y,z).$$

Beräkna flödet av u genom K.

Lösn.



∂K = Sfär av radie $\sqrt{3}$

$\cup$ Sfär av radie $\sqrt{2}$

$=: S_3 \cup S_2$

$$\begin{cases} N_3(x,y,z) = \dfrac{(x,y,z)}{\sqrt{x^2+y^2+z^2}} = \dfrac{\vec{r}}{r} & \forall (x,y,z) \in S_3 \\ N_2(x,y,z) = -\dfrac{(x,y,z)}{\sqrt{x^2+y^2+z^2}} = -\dfrac{\vec{r}}{r} & \forall (x,y,z) \in S_2. \end{cases}$$

OBS: $u(\vec{r}) = \dfrac{\vec{r}}{r^2}$.

$$\Longrightarrow \iint\limits_{\partial K} \langle u, N_{\partial K} \rangle dA = \iint\limits_{S_3} \langle u, N_3 \rangle dA + \iint\limits_{S_2} \langle u, N_2 \rangle dA$$

$$\underset{\text{(sfäriska koord.)}}{=} \int_0^{2\pi}\int_0^{\pi} \left\langle \frac{\vec{r}}{r^2}, \frac{\vec{r}}{r} \right\rangle r^2 \sin\varphi \, d\varphi \, d\theta \Big|_{r=\sqrt{3}}$$

$$+ \int_0^{2\pi}\int_0^{\pi} \left\langle \frac{\vec{r}}{r^2}, \frac{-\vec{r}}{r} \right\rangle r^2 \sin\varphi \, d\varphi \, d\theta \Big|_{r=\sqrt{2}}$$

$$= \left\{ \left\langle \frac{\vec{r}}{r}, \frac{\vec{r}}{r} \right\rangle = \left\| \frac{\vec{r}}{r} \right\|^2 = 1 \right\}$$

$$= 2\pi r \underbrace{\int_0^{\pi} \sin\varphi \, d\varphi}_{=2} \Big|_{r=\sqrt{3}} - 2\pi r \int_0^{\pi} \sin\varphi \, d\varphi \Big|_{r=\sqrt{2}}$$

$$= 4\pi\sqrt{3} - 4\pi\sqrt{2} = 4\pi(\sqrt{3} - \sqrt{2}).$$

PAUS till 16:18!

10.35: $u(x,y,z) = (x-3y+z^2,\ 2x-y^2+z,\ x^2+y^2-2z^2)$.

Beräkna $(= (u_1, u_2, u_3))$

a) $\operatorname{div} u = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}$

$= 1 - 2y - 4z$

b) $\operatorname{rot} u = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \times \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z} \\ \frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x} \\ \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \end{pmatrix}$

$= \begin{pmatrix} 2y - 1 \\ 2z - 2x \\ 2 - (-3) \end{pmatrix} = \begin{pmatrix} 2y-1 \\ 2(z-x) \\ 5 \end{pmatrix}$.

c) ?? ?? $\operatorname{grad} u =$ ej väldefinierat!!! $\neq$

d) $\operatorname{grad} \operatorname{div} u = \operatorname{grad}(1-2y-4z) = \begin{pmatrix} 0 \\ -2 \\ -4 \end{pmatrix} = -2\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$.

e) $\operatorname{rot} \operatorname{rot} u = \operatorname{rot} \begin{pmatrix} 2y-1 \\ 2(z-x) \\ 5 \end{pmatrix}$

$= \begin{pmatrix} 0 - 2 \\ 0 - 0 \\ -2 - 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -4 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

10.54: Beräkna

$$\int_\gamma (3x + z\cos(yz))\,dy + (y - 2x + y\cos(yz))\,dz,$$

där $\gamma = \partial K$, $K \subset \{(x,y,z) : 2x+y+2z = 5\}$ kompakt med $\operatorname{Area}(K) = 3$.

Lösn. Låt $F(x,y,z) = (0,\ 3x + z\cos(yz),\ y - 2x + y\cos(yz))$.

Stokes: $\int_{\partial K} \langle F, dr\rangle = \iint_K \langle \operatorname{rot} F, N_K\rangle\, dA$.

1) $N_K = \frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ $\quad\left(2x+y+2z = 5 \Leftrightarrow \left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right\rangle = 5 \right)$

2) $$\operatorname{rot} F = \begin{pmatrix} 1 + \cos(yz) - yz\sin(yz) - (\cos(yz) - zy\sin(yz)) \\ 0 - (-2) \\ 3 - 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\Rightarrow \int_{\partial K} \langle F, dr \rangle = \iint_K \langle \operatorname{rot} F, N_K \rangle \, dA$$

$$= \iint_K \left\langle \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right\rangle dA$$

$$= \langle \text{-} \text{''} \text{-} \rangle \cdot \underbrace{\iint_K dA}_{= \operatorname{Area}(K)}$$

$$= \frac{1}{3}(2 + 2 + 3 \cdot 2) \underbrace{\operatorname{Area}(K)}_{= 3}$$

$$= 2 + 2 + 3 \cdot 2 = \underline{\underline{10}}.$$