MVE035/600 Gammal tenta 11/3-17

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1. $f(x,y) = x^2+y^2+\frac{2}{xy}$, $xy \neq 0$.

a) $\nabla f = 0 \iff \begin{cases} 2x - \frac{2}{x^2y} = 0 \\ 2y - \frac{2}{xy^2} = 0 \end{cases}$

$\iff \begin{cases} x^3y = 1 \\ xy^3 = 1 \end{cases}$

$\overset{xy\neq 0}{\implies} x^2 = y^2 \iff x = \pm y$

1) $x = y \implies x^4 = 1 \implies x^2 = 1 \implies x = \pm 1$, $y = \pm 1$

2) $x = -y \implies \cancel{x^4 = -1}$

Stationära pkter: $(1,1)$, $(-1,-1)$.

Karakterer: $Hf(x,y) = \begin{pmatrix} 2+\frac{4}{x^3y} & \frac{2}{x^2y^2} \\ \frac{2}{x^2y^2} & 2+\frac{4}{xy^3} \end{pmatrix}$

$\implies \det Hf(\pm1,\pm1) = \det\begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix}$

$= 32 > 0$

$\implies$ pkterna är minima!

b) $f(2,-1) = 4$, $f_x(2,-1) = \frac{9}{2}$, $f_y(2,-1) = -3$,

$f_{xx}(2,-1) = \frac{3}{2}$, $f_{xy}(2,-1) = \frac{1}{2}$, $f_{yy}(2,-1) = 0$.

$\implies T^{(2)}_{(2,-1)} f(h,k) = 4 + \frac{9}{2}h - 3k + \frac{3}{4}h^2 + \frac{1}{2}hk$.

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2. $f(x,y,z) = x^2+y^2-z^3$

a) $\nabla F(1,3,2) = \begin{pmatrix}2x\\2y\\-3z^2\end{pmatrix}\Bigg|_{(1,3,2)} = \begin{pmatrix}2\\6\\-12\end{pmatrix} = 2\begin{pmatrix}1\\3\\-6\end{pmatrix}$

Tangentplan: $\left\langle \nabla F(1,3,2), \begin{pmatrix}x-1\\y-3\\z-2\end{pmatrix}\right\rangle = 0$

$$\Updownarrow$$

$$x+3y-6z=-2$$

$$\Updownarrow$$

$$-x-3y+6z=2$$

b) $\frac{\partial F}{\partial z}(1,3) = -12 \neq 0 \xRightarrow[\text{Funk.}]{\text{Imp.}} \exists f:\mathbb{R}^2\to\mathbb{R}$ s.a. $z=f(x,y)$ ger $F=2$, nära $(1,3)$.

Riktningsderivatan:

$$\nabla f(1,3) \underset{\text{Imp. Funk.}}{=} \begin{pmatrix}-F_x/F_z\\-F_y/F_z\end{pmatrix}\Bigg|_{(1,3,2)} = \dots = \left(\frac{1}{6}, \frac{1}{2}\right)$$

i riktningen $u=(2,5)-(1,3)=(1,2)$

$$\frac{\partial f}{\partial u}(1,3) = \langle \nabla f(1,3), \hat{u}\rangle = \frac{1}{\sqrt{5}}\left\langle \begin{pmatrix}1/6\\1/2\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}\right\rangle$$

$$= \underline{\underline{\frac{7}{6\sqrt{5}}}}$$

c) $g_1(x,y,z)=x^2+y^2-z^2$, $g_2(x,y,z)=x-2z-3$.

Maximera/minimera F givet $g_1=g_2=0$.

$$0 \overset{?}{=} \det\left(\nabla F\ \nabla g_1\ \nabla g_2\right) = \det\begin{pmatrix}2x&2x&1\\2y&2y&0\\-3z^2&-2z&-2\end{pmatrix}$$

$$=\det\begin{pmatrix}2x&2x&1\\2y&2y&0\\-3z^2+4x&-2z+4x&0\end{pmatrix} = 2y\det\begin{pmatrix}1&1\\-3z^2+4x&-2z+4x\end{pmatrix}$$

$$= -2yz(3z+2)$$

<u>3 Fall:</u>

1) $y = 0 \implies 0 = g_1(x,0,z) = x^2 - z^2, \quad 0 = g_2(x,0,z) = x - 2z - 3$

$$x = \pm z, \quad x \pm 2x = 3 \implies \underset{(+)}{x = 1} \;\vee\; \underset{(-)}{x = -3}$$

$$z = 1 \;\vee\; z = 3$$

2) $z = 0 \implies 0 = g_2(x,y,0) = x - 3, \quad x = 3$

$$0 = g_1(x,y,0) = x^2 + y^2 = 9 + y^2$$

$$\implies y^2 \cancel{=} -9$$

3) $z = -\frac{2}{3}: \quad 0 = g_2(x,y,-\frac{2}{3}) = x + \frac{4}{3} - 3$

$$\implies x = \frac{5}{3}$$

$$\implies 0 = g_1(\tfrac{5}{3}, y, \tfrac{2}{3}) = \frac{21}{9} \cancel{+} y^2$$

<u>OBD:</u> $F(3,0,-3) = 36, \quad F(1,0,1) = 2$

max   min

<u>3.</u> $D = \{(x,y) : 0 \le x \le 1, \sqrt{x} \le y \le 1\}$

$$= \{(x,y) : 0 \le y \le 1, \; 0 \le x \le y^2\}$$

$$\implies \iint_D e^{y^3}\,dx\,dy = \int_0^1 \int_0^{y^2} dx \cdot e^{y^3} dy$$

$$= \frac{1}{3}\int_0^1 3y^2 e^{y^3} dy$$

$$= \frac{1}{3}\int_0^1 e^u du = \frac{e-1}{3}.$$

DAUS till [illegible]!

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4. $r(s,t) = \left(s + \frac{t^3}{3} - 1,\ s - \frac{t^3}{3} - 1,\ st^2\right)$, $\quad 0 \le s \le 2$, $0 \le t \le 3$

a) $$\text{Area}(Y) = \int_0^2 \int_0^3 \underbrace{\left\| \frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t} \right\|}_{=: f(s,t)} ds\,dt$$

$$\frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t} = \begin{pmatrix} 1 \\ 1 \\ t^2 \end{pmatrix} \times \begin{pmatrix} t^2 \\ -t^2 \\ 2st \end{pmatrix} = t \begin{pmatrix} 2s + t^3 \\ t^3 - 2s \\ -2t \end{pmatrix}$$

$$\Rightarrow f(s,t) = t\sqrt{(2s+t^3)^2 + (2s-t^3)^2 + 4t^2}$$
$$= t\sqrt{8s^2 + 2t^6 + 4t^2}.$$

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b) $\gamma(t) = r(1,t) = \left(\frac{t^3}{3}, -\frac{t^3}{3}, t^2\right)$, $0 \le t \le 3$,

$\gamma'(t) = (t^2, t^2, 2t)$, $\quad \|\gamma'(t)\| = t\sqrt{2t^2+4}$

$$\Rightarrow \ell(\gamma) := \int_0^3 \|\gamma'(t)\|\,dt = \frac{\sqrt{2}}{2}\int_0^3 \sqrt{t^2+2}\cdot 2t\,dt$$
$$= \frac{1}{\sqrt{2}}\int_0^9 \sqrt{u+2}\,du = \frac{1}{\sqrt{2}}\int_2^{11}\sqrt{v}\,dv$$
$$= \frac{2}{3\sqrt{2}}\left(11^{3/2} - 2^{3/2}\right)$$
$$= \frac{1}{3}\left(11\sqrt{22} - 4\right).$$

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c) $F(x,y,z) = \left(e^{yz} - \pi y^2 \sin(\pi x),\ xz e^{yz} + 2y\cos(\pi x),\ xy e^{yz} + 2z\right)$

Potentialfält? $U(x,y,z) = x e^{yz} + y^2\cos(\pi x) + z^2$ är en potential till F!

$$\Rightarrow \int_\gamma \langle F, d\gamma\rangle = U(\gamma(3)) - U(\gamma(0))$$
$$= \ldots = 9e^{-81} \approx 0.$$

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5. $K = \{(x,y,z) : 1+\sqrt{x^2+y^2} \le z \le \sqrt{25-x^2-y^2}\}$

$= \{(r,\theta,z) : 1+r \le z \le \sqrt{25-r^2}\}$

(cyl.)

$F(x,y,z) = ((x-y)z,\ x^2+yz,\ z^2+y)$

b) $\operatorname{div} F = z + z + 2z = 4z$

Gauss: $\iint_{\partial K} \langle F, N_K\rangle\, dA = \iiint_K \operatorname{div} F\, dV$

$= 4\iint_D \int_{1+r}^{\sqrt{25-r^2}} z\, dz\, d\theta\, dr$, $\quad D = \{(r,\theta) : 1+r \le \sqrt{25-r^2}\}$

D: $(1+r)^2 \le 25 - r^2 \iff r^2 + r \le 12$

$r^2 + r = 12 \Rightarrow (r+\frac{1}{2})^2 = \frac{49}{4}$

$r = \pm\frac{7}{2} - \frac{1}{2} = \begin{cases} 3 & \text{om } + \\ \cancel{-4 \text{ om } -} \end{cases}$

$\Rightarrow D$: $0 \le r \le 3$ (disk av radie 3)

$= 2\int_0^3 \int_0^{2\pi} (25 - r^2 - (1+r)^2)\, r\, d\theta\, dr$

$= 4\pi \int_0^3 (24 - 2r^2 - 2r)\, dr = \dots = 198\pi.$

c) Enl. b) $\iiint_K z\, dV = \frac{41\pi}{12}$,

$\iiint_K dV = \iint_D \int_{1+r}^{\sqrt{25-r^2}} dz\, dA_D = \int_0^3 \int_0^{2\pi} (\sqrt{25-r^2} - 1 - r)\, r\, d\theta\, dr$

$= 2\pi\left(\frac{1}{2}\int_0^3 \sqrt{25-r^2}\cdot 2r\, dr - \int_0^3 r\, dr - \int_0^3 r^2\, dr\right)$

$= 2\pi\left(\frac{1}{2}\underbrace{\int_0^9 \sqrt{25-t}\, dt}_{} - \frac{9}{2} - 9\right)$

$= \int^{25} \sqrt{u}\, du = \frac{2}{3}(25\cdot 5 - 16\cdot 4)$

$$= \frac{122}{3}$$

$$= 2\pi\left(\frac{61}{3} - \frac{9}{2} - 9\right) = \frac{82\pi}{6} = \frac{41\pi}{3}$$

$$\Longrightarrow m(K)_z = \left(\frac{198\pi}{4}\right) \Big/ \left(\frac{41\pi}{3}\right) = 297/82$$

D: D är symmetrisk i (x,y)-planet så $m(K)_x = m(K)_y = 0$.

<u>Svar</u>: $m(K) = \left(0, 0, \frac{297}{82}\right)$.

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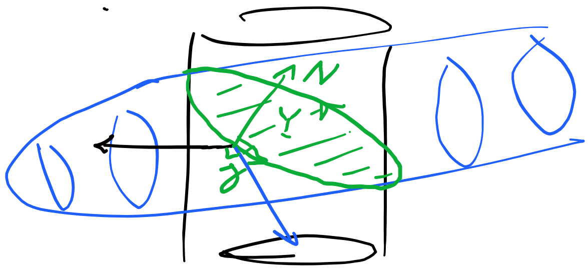
<u>6.</u> $F(x,y,z) = (y, -x, z^2 - xy)$,

$$\gamma = \{(x,y,z) : z = y^2,\ x^2+y^2 = 9\}$$

<u>Stokes</u>: $\displaystyle\int_\gamma \langle F, dr\rangle = \iint_Y \langle \operatorname{rot} F, N\rangle \, dA$

? ?

$$N = \operatorname{grad}(z - y^2) = \begin{pmatrix} 0 \\ 2y \\ -1 \end{pmatrix}$$



Y : $z = y^2$, $x^2 + y^2 \le 9$

$$\operatorname{rot} F = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} y \\ -x \\ z^2 - xy \end{pmatrix} = \begin{pmatrix} -x - 0 \\ 0 - (-y) \\ -1 - 1 \end{pmatrix} = \begin{pmatrix} -x \\ y \\ -2 \end{pmatrix}$$

$$\Longrightarrow \iint_Y \langle \operatorname{rot} F, N\rangle \, dA = -\iint_{x^2+y^2\le 9} (2y^2 + 2)\, dx\, dy = -2\int_0^{2\pi}\int_0^3 (r^2 \sin^2\theta + 1)\, r\, dr\, d\theta$$

$$= \ldots = -\frac{117\pi}{2}.$$