MVE035/600 Gammal tenta 7/6-17

Wednesday, 10 March 2021 15:14

1. $f(x,y) = \dfrac{x+y}{1+x^2+y^2}$

a) $$\nabla f(3,1) = \left.\begin{pmatrix} \dfrac{1-x^2-2xy+y^2}{(1+x^2+y^2)^2} \\ \dfrac{1+x^2-2xy-y^2}{(1+x^2+y^2)^2} \end{pmatrix}\right|_{(3,1)}$$

$$= \frac{1}{121}\begin{pmatrix} -13 \\ 3 \end{pmatrix}.$$

Riktningsderivata: $u = (4,4)-(3,1) = (1,3)$, $\|u\| = \sqrt{10}$

$$\Rightarrow \frac{\partial f}{\partial u}(3,1) = \langle \nabla f(3,1), \hat{u} \rangle = \frac{1}{121\cdot\sqrt{10}} \left\langle \begin{pmatrix} -13 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\rangle$$

$$= \frac{-4}{121\sqrt{10}}$$

Tangentplan: $$\left\langle \begin{pmatrix} \nabla f(3,1) \\ -1 \end{pmatrix}, \begin{pmatrix} x-3 \\ y-1 \\ z-4/11 \end{pmatrix} \right\rangle = 0$$

$$\iff 13x - 3y + 121z = 80.$$

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b) $f(x,y) = \dfrac{x+y}{1+x^2+y^2}$ är icke-negativ i D

med $f(0,0) = 0$ och $\lim\limits_{\substack{(x,y)\to\infty \\ i\ D}} f(x,y) = 0$,

så bilden $f(D)$ är kompakt ur kontinuitet

och $\exists$ max, min av f.

Minimum är 0.

Maximum:

Inre pkter: $\nabla f = 0 \iff \begin{cases} 1-x^2-2xy+y^2 = 0 \\ 1+x^2-2xy-y^2 = 0 \end{cases}$

$$\overset{(1,-1)}{\Longleftrightarrow} \begin{cases} 2-4xy=0 \\ -2(x^2-y^2)=0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} 2-4x^2=0 \\ x=y \end{cases} \Longrightarrow x=\frac{1}{\sqrt{2}}=y$$

$$f\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)=\frac{\sqrt{2}}{1+\frac{1}{2}+\frac{1}{2}}=\frac{1}{\sqrt{2}}.$$

Maximum?

Randpunkter: $f(x,y)=f(y,x)$, så räcker att minimera $g(x):=f(x,0)$, $x\geq 0$.

$$=\frac{x}{1+x^2}$$

$$g'(x)=\frac{1-x^2}{1+x^2}=0 \Longrightarrow x=1.$$

$$g(1)=\frac{1}{1+1^2}=\frac{1}{2}<\frac{1}{\sqrt{2}}.$$

Svar) $\max\limits_{(x,y)\in D} f(x,y)=\frac{1}{\sqrt{2}}$, $\min\limits_{(x,y)\in D} f(x,y)=0$.

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3. a) $$\int_0^2\int_0^{4-x^2}\frac{x e^{2y}}{4-y}\,dy\,dx=\left\{D:=\{(x,y): 0\leq x\leq 2,\ 0\leq y\leq 4-x^2\}\right.$$

$0\leq x\leq\sqrt{4-y}$

$$=\{(x,y): 0\leq y\leq 4,\ 0\leq x\leq\sqrt{4-y}\}\Big\}$$

$$=\int_0^4\int_0^{\sqrt{4-y}}x\,dx\,\frac{e^{2y}}{4-y}\,dy=\frac{1}{2}\int_0^4 \cancel{(4-y)}\,\frac{e^{2y}}{\cancel{4-y}}\,dy$$

$$=\frac{1}{4}\left[e^{2y}\right]_0^4=\frac{e^8-1}{4}.$$

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4. a) $F(x,y,z)=\left(2\cos y,\ \frac{1}{4}-2x\sin y,\ \frac{1}{z}\right)$

$\gamma: (0,2,1)\longrightarrow(1,\pi,2)$

Är F konservativt?

$$\left(\int F_1\,dx=2x\cos y+f_1(\cancel{x},y,z)\right.$$

$$\begin{cases} \int F_2\,dy = \log y + 2x\cos y + f_2(x,\cancel{y},z) \\ \int F_3\,dz = \log z + f_3(x,y,\cancel{z}) \end{cases}$$

$\rightsquigarrow U(x,y,z) = 2x\cos y + \log(yz)$ är en potential till F.

$$\Rightarrow \int_\gamma \langle F, dr\rangle = U(1,\pi,2) - U(0,2,1)$$

$$= 2\cos\pi + \log(2\pi) - \log(2)$$

$$= -2 + \log\left(\frac{2\pi}{2}\right) = \log\pi - 2.$$

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5. a) $F(x,y) = (e^x + 6xy,\ 8x^2 + \sin(y^2))$

Green: $\displaystyle\int_{\partial D} \langle F, dr\rangle = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$

$$= \left\{ \frac{\partial F_2}{\partial x} = 16x,\ \frac{\partial F_1}{\partial y} = 6x \right\}$$

$$= 10 \int_0^{\pi/2}\int_1^3 r\cos\theta \cdot r\,dr\,d\theta$$

$$= 10 \cdot \int_0^{\pi/2} \cos\theta\, d\theta \cdot \int_1^3 r^2\,dr$$

$$= 10 \cdot 1 \cdot \frac{1}{3}(27-1)$$

$$= \frac{260}{3}.$$

b) $D = \{(x,y) : x^{2/3} + y^{2/3} \le 1\}$,

$\partial D = \{(\cos^3 t, \sin^3 t) : t \in [0, 2\pi]\}$.

Green: $\displaystyle \text{Area}(D) = \iint_D \left(\frac{1}{2} - \left(-\frac{1}{2}\right)\right) dx\,dy$

$$= \frac{1}{2}\int_{\partial D} -y\,dx + x\,dy$$

$$= \{ \gamma(t) = (\cos^3 t, \sin^3 t),$$
$$\gamma'(t) = (-3\sin t\cos^2 t,\ 3\cos t\sin^2 t) \}$$

$$= \frac{1}{2}\int_0^{2\pi} \left\langle \begin{pmatrix} -\sin^3 t \\ \cos^3 t \end{pmatrix}, \begin{pmatrix} -3\sin t\cos^2 t \\ 3\cos t\sin^2 t \end{pmatrix} \right\rangle dt$$

$$\iint_D (1-0)\,dx\,dy$$

$$= \int_{\partial D} 0\,dx + x\,dy$$

$$= \int^{2\pi} \left\langle \begin{pmatrix} 0 \\ \cos^3 t \end{pmatrix}, \begin{pmatrix} - \\ 3\cos t\sin^2 t \end{pmatrix} \right\rangle dt$$

$$= 3\int_0^{2\pi} \cos^4 t \sin^2 t \, dt$$

$$= 3\int_0^{2\pi} \cos^3 t \cdot \sin^2 t \cdot \cos t \, dt$$

$$= \{ u = \sin t, \ du = \cos t \, dt \}$$

$$= 3\int_0^1 u^2 \cdot (1-u^2)^{3/2} \, du$$

$$\text{Area}(D) = \int_{\partial D} -\lambda y \, dx + (1-\lambda) x \, dy \quad \forall \lambda \in [0,1]$$

$$= \frac{3}{2}\int_0^{2\pi} (\sin^2 t \cdot \sin^2 t \cos^2 t + \cos^2 t \cdot \cos^2 t \sin^2 t) \, dt$$

$$= \frac{3}{2}\int_0^{2\pi} \sin^2 t \cos^2 t \, dt$$

$$= \frac{3}{8}\int_0^{2\pi} \sin^2 2t \, dt$$

$$= \frac{3}{16}\int_0^{2\pi} (1 - \cancel{\cos 4t}) \, dt$$

$$= \frac{3\pi}{8}$$

PAUS till 16:12!

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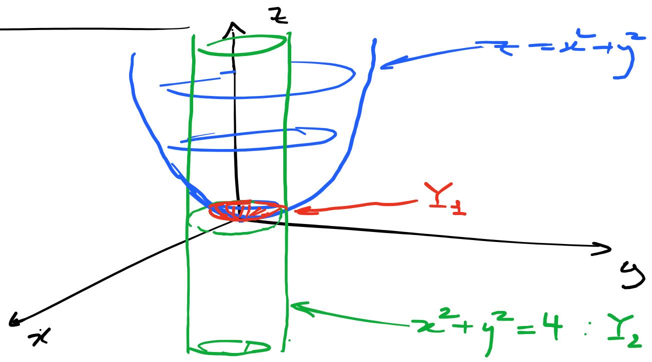
6. a)



$$\text{Area}(Y_1) = \iint_{Y_1} \|N_{Y_1}\| \, dA = \iint_{x^2+y^2\le 4} \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} \, dx \, dy$$

$(Y_1 : z = x^2+y^2)$

$$= \iint_{x^2+y^2 \le 4} \sqrt{4(x^2+y^2)+1} \, dx \, dy$$

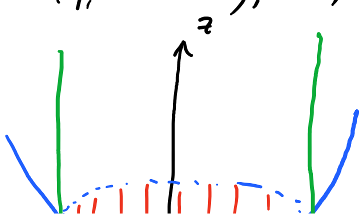
$$= \int_0^{2\pi}\int_0^2 \sqrt{4r^2+1} \cdot r \, dr \, d\theta$$

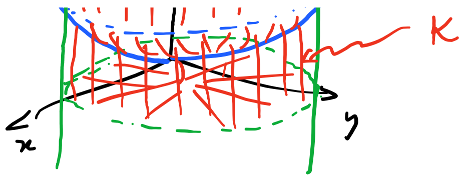
$$= \frac{2\pi}{8}\int_1^{17} \sqrt{u} \, du = \frac{2\pi}{8}\left[\frac{2}{3}u^{3/2}\right]_1^{17}$$

$\left(\begin{array}{l} u = 4r^2+1 \\ du = 8r \, dr \end{array} \right)$

$$= \frac{\pi}{6}(17\sqrt{17} - 1).$$

b) $F(x,y,z) = (y, -x, z^2)$





$\operatorname{div} F(x,y,z) = 2z,$

<u>Gauss:</u> $\iint_{\partial K} \langle F, N\rangle\, dA = \iiint_K \operatorname{div} F\, dV$

$$= \iint_{x^2+y^2\le 4} \int_0^{x^2+y^2} 2z\, dz\, dA$$

$$= \iint_{x^2+y^2\le 4} (x^2+y^2)^2\, dx\, dy$$

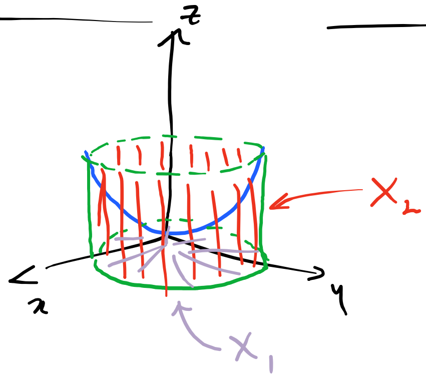
$$= \int_0^{2\pi}\int_0^2 r^4 \cdot r\, dr\, d\theta$$

(polare koord.)

$$= 2\pi \cdot \frac{1}{6}\cdot (2^6 - 0)$$

$$= \frac{64\pi}{3}.$$

$$\iint_{Y_1} \langle F, N\rangle\, dA \underset{\text{Gauss}}{=} \iiint_K \operatorname{div} F\, dV - \iint_{\partial K\setminus Y_1} \langle F, N\rangle\, dA$$



$\partial K \setminus Y_1 = X_1 \cup X_2,$

1) $\langle F, N_{X_1}\rangle = \left\langle \begin{pmatrix} y \\ -x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ * \end{pmatrix} \right\rangle$

$= 0$

2) $\langle F, N_{X_2}\rangle = \left\langle \begin{pmatrix} y \\ -x \\ z^2 \end{pmatrix}, \begin{pmatrix} \hat{x} \\ \hat{y} \\ 0 \end{pmatrix} \right\rangle$

$\left(\hat{x} = \frac{1}{\sqrt{x^2+y^2}}\cdot x\right)$ etc. $\quad = y\hat{x} - x\hat{y} = 0$

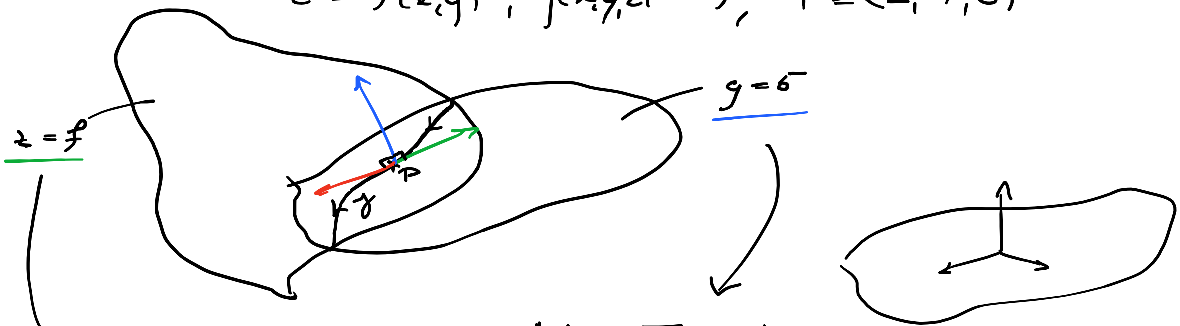
$$= \iiint_K \operatorname{div} F\, dV = \frac{64\pi}{3}.$$

10/3-18 :

1. b) $f(x,y) = 3x\log(y+2) + x - 2y^2$

$g(x,y,z) = x + ye^z + x^2y^2$,

$z = f(x,y)$, $g(x,y,z) = 5$, $P = (2,-1,0)$



$N_f = \begin{pmatrix} -f_x \\ -f_y \\ 1 \end{pmatrix}(P)$ $\qquad$ $\underline{N_g = \nabla g(P)}$

$\rightsquigarrow \gamma'(t_0) = N_f(P) \times N_g(P)$

$\gamma(t_0) = P$