Basic financial concepts

for the course Financial derivative and PDE's

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The main topic of the course "Financial derivatives and PDE's is the theoretical valuation of financial derivatives based on the arbitrage-free principle and using methods from stochastic calculus and partial differential equations. The purpose of this text is to introduce the basic financial terminology used in the rest of the course. More advanced financial concepts (e.g., forwards, futures, swaps and coupon bonds) are discussed in the lecture notes of the course. This text is an abridged version of Chapter 1 in [1].

Financial assets

The term **asset** may be used to identify any resource capable of producing value and which, under specific legal terms, can be bought and sold (i.e., converted into cash). Assets may be tangible (e.g., lands, buildings, commodities, etc.) or intangible (e.g., patents, copyrights, stocks, etc.). Assets are also divided into **real assets**, i.e, assets whose value is derived by an intrinsic property (e.g., tangible assets), and **financial assets**, such as stocks, options, bonds, etc., whose value is instead derived from a contractual claim on the income generated by another (possibly real) asset. For example, upon holding shares of the Volvo stock (a financial asset), we can make a profit from the production and sale of cars even if we do not own an auto plant (a real asset). As we consider only financial assets in this text, the terms "asset" and "financial asset" will be henceforth used interchangeably.

Price

The **price** of a financial asset is the value, measured in some units of currency (e.g. dollars), at which the **buyer** and the **seller** agree to exchange ownership of the asset. The price is chosen by the two parties as a result of some kind of "negotiation". More precisely, the **ask price** is the minimum price at which the seller is willing to sell the asset, while the **bid price** is the maximum price that the buyer is willing to pay for the asset. A **transaction** occurs when the bid price of a buyer matches the ask price of a seller, in which case the exchange of the asset takes place at the corresponding price.

A generic financial asset will be denoted by $\mathcal{U}$ and its price at time t by $\Pi^{\mathcal{U}}(t)$. Prices are generally positive, although some financial assets (e.g., forward contracts) have zero price.

The asset price refers to the price per **share** of the asset, where "share" stands for the minimum amount of an asset that can be traded. All prices in this text are given in a fixed currency, which is however left unspecified.

Markets

Financial assets can be traded in **exchange** markets or **over the counter (OTC)**. In the former case all trades are subject to a common regulation, while in the latter the trading conditions are more flexible and, to a certain degree, can be agreed upon by the individual traders. The same asset can often be traded both in an exchange market and OTC, usually for a different price. The advantage of trading in regularized exchange markets is the higher level of transparency and protection offered by standardized contracts.

Examples of official exchange markets, respectively of stocks and options, are the Nasdaq market and the Chicago Board of Options Exchange (CBOE); currencies are example of financial assets which are traded only OTC (**Forex market**).

A **market maker** is large investment company that continuously quotes both an ask price and a bid price for immediate purchase/sell of an asset, thereby ensuring markets **liquidity**. The difference between the bid and the ask price of an asset quoted by a market maker is also called the **bid-ask spread** of the asset.

Any transaction in the market is subject to **transaction costs** (e.g., exchange fees) and **transaction delays** (trading in real markets is not instantaneous).

Buyers and sellers of assets in a market will be called **investors** or **agents**.

Long and short position

An investor is said to **short-sell** N shares of an asset if the investor borrows the shares from a third party and then sell them immediately on the market. The reason for short-selling an asset is the expectation that the price of the asset will decrease in the future. In fact, suppose that N shares of an asset $\mathcal{U}$ are short-sold at time $t = 0$ for the price $\Pi^{\mathcal{U}}(0)$ and let $T > 0$. If $\Pi^{\mathcal{U}}(T) < \Pi^{\mathcal{U}}(0)$, then upon re-purchasing the N shares at time T, and returning them to the lender, the short-seller will make the profit $N(\Pi^{\mathcal{U}}(0) - \Pi^{\mathcal{U}}(T))$.

An investor is said to have a **long position** on an asset if the investor owns the asset and will therefore profit from an increase of its price. Conversely, the investor is said to have a **short position** on the asset if the investor will profit from a decrease of its value, as it happens for instance when the investor is short-selling the asset.

Stocks and dividends

The **capital stock** of a company is the part of the company equity capital that is made publicly available for trading. Stocks are most commonly traded in official exchange markets. For instance, over 300 company stocks are traded in the Stockholm exchange market. The price per share of a generic stock at time t will be denoted by $S(t)$.

A stock may occasionally pay a **dividend** to its shareholders, usually in the form of a cash deposit. The amount (per share) of the dividend and its **payment date** must be declared in advance (**announcement date**). The **ex-dividend date** is the first day before the payment date (usually a few days before it) at which buying the stock does not entitle to the dividend. An investor who buys the stock prior to the ex-dividend day and holds it until the ex-dividend day is entitled to the dividend, even if the investor does not own the stock at the payment day. At the ex-dividend day, the price of the stock often (but not always) drops of roughly the same amount paid by the dividend.

Market index and ETF's

A market **index** is a weighted average of the value of a collection of assets traded in one or more exchange markets. For example, S&P500 (Standard and Poor 500) measures the average value of 500 stocks traded at the New York stock exchange (NYSE) and NASDAQ markets. Market indexes can be regarded themselves as tradable assets. More precisely an **ETF** (Exchange Traded Fund) on a market index is a financial asset whose value tracks exactly the value of the market index (or a given fraction thereof). Hence one share of an ETF on S&P500 will increase its value of 1% in one day if during that day S&P 500 has gained 1%. An **inverse ETF** however will in the same example decrease its value of 1%. Thus ETF's give investors the possibility to speculate whether the market will gain or loose value in the future.

Portfolio position and portfolio process

Consider an agent that invests on N assets $\mathcal{U}_1, \dots, \mathcal{U}_N$ during the time interval $[0, T]$. Assume that the agent trades on a_1 shares of the asset $\mathcal{U}_1$, a_2 shares of the asset $\mathcal{U}_2$,..., a_N shares of the asset $\mathcal{U}_N$. Here $a_i \in \mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$, where $a_i < 0$ means that the investor has a short position on the asset $\mathcal{U}_i$, while $a_i > 0$ means that the investor has a long position on the asset $\mathcal{U}_i$ (the reason for this interpretation will become soon clear). The vector $\mathcal{A} = (a_1, a_2, \dots, a_N) \in \mathbb{Z}^N$ is called a **portfolio position**, or simply a portfolio. The **value** of the portfolio at time t is given by

$$V_{\mathcal{A}}(t) = \sum_{i=1}^{N} a_i \Pi^{\mathcal{U}_i}(t), \quad t \in [0, T], \tag{1}$$

where $\Pi^{\mathcal{U}_i}(t)$ denotes the price of the asset $\mathcal{U}_i$ at time t. The value of the portfolio measures the wealth of the investor: the higher is $V(t)$, the "richer" is the investor at time t. It follows that when the price of the asset $\mathcal{U}_i$ increases, the value of the portfolio increases if $a_i > 0$ and decreases if $a_i < 0$, which explains why $a_i > 0$ corresponds to a long position on the asset $\mathcal{U}_i$ and $a_i < 0$ to a short position. Portfolios can be added by using the linear structure on $\mathbb{Z}^N$, namely if $\mathcal{A}, \mathcal{B} \in \mathbb{Z}^N$, $\mathcal{A} = (a_1, \dots, a_N)$, $\mathcal{B} = (b_1, \dots, b_N)$ are two portfolios and $\alpha, \beta \in \mathbb{Z}$, then $\mathcal{C} = \alpha\mathcal{A} + \beta\mathcal{B}$ is the portfolio $\mathcal{C} = (\alpha a_1 + \beta b_1, \dots, \alpha a_N + \beta b_N)$, whose value is given by $V_{\mathcal{C}}(t) = \alpha V_{\mathcal{A}}(t) + \beta V_{\mathcal{B}}(t)$.

In the definition of portfolio position and portfolio value given above, the investor keeps the same number of shares of each asset during the whole time interval $[0, T]$. Suppose now that the investor changes the position on the assets at some times $t_1, \dots, t_{M-1}$, where

$$0 = t_0 < t_1 < t_2 < \dots < t_{M-1} < t_M = T;$$

for simplicity we assume that at each time $t_1, \dots, t_{M-1}$ the change in the portfolio position occurs instantaneously. Let $\mathcal{A}_0$ denote the initial (at time $t = t_0 = 0$) portfolio position of the investor and $\mathcal{A}_j$ denote the portfolio position of the investor in the interval of time $(t_{j-1}, t_j]$, $j = 1, \dots, M$. As positions hold for one instance of time only are clearly meaningless, we may assume that $\mathcal{A}_0 = \mathcal{A}_1$, i.e., $\mathcal{A}_1$ is the portfolio position in the closed interval $[0, t_1]$. The vector $(\mathcal{A}_1, \dots, \mathcal{A}_M)$ is called a **portfolio process**. Denoting by a_{ij} the number of shares of the asset i in the portfolio $\mathcal{A}_j$, a portfolio process is equivalent to the $N \times M$ matrix $A = (a_{ij})$, $i = 1, \dots, N$, $j = 1, \dots, M$. The value $V(t)$ of the portfolio process at time t is given by the value of the corresponding portfolio position at time t as defined by (1), that is

$$V(t) = \begin{cases} V_{\mathcal{A}_1}(t) = \sum_{i=1}^{N} a_{i1} \Pi^{\mathcal{U}_i}(t), & \text{for } t \in [0, t_1] \\ V_{\mathcal{A}_2}(t) = \sum_{i=1}^{N} a_{i2} \Pi^{\mathcal{U}_i}(t), & \text{for } t \in (t_1, t_2] \\ \vdots & \vdots \\ V_{\mathcal{A}_M}(t) = \sum_{i=1}^{N} a_{iM} \Pi^{\mathcal{U}_i}(t), & \text{for } t \in (t_{M-1}, t_M] \end{cases}.$$

The initial value $V(0) = V_{\mathcal{A}_0} = V_{\mathcal{A}_1}(0)$ of the portfolio, when it is positive, is called the **initial wealth** of the investor.

A portfolio process is said to be **self-financing** if the portfolio assets pay no dividends and if no cash is ever withdrawn or infused in the portfolio. For example, let $\mathcal{U}_1$, $\mathcal{U}_2$, $\mathcal{U}_3$ be non-dividend paying assets in the interval $[0, T]$. Suppose that at time $t_0 = 0$ the investor is short 400 shares on the asset $\mathcal{U}_1$, long 200 shares on the asset $\mathcal{U}_2$ and long 100 shares on the asset $\mathcal{U}_3$. This corresponds to the portfolio

$$\mathcal{A}_0 = (-400, 200, 100),$$

whose value is

$$V_{\mathcal{A}_0} = -400\,\Pi^{\mathcal{U}_1}(0) + 200\,\Pi^{\mathcal{U}_2}(0) + 100\,\Pi^{\mathcal{U}_3}(0).$$

If this value is positive, the investor needs an initial wealth to set up this portfolio position: the income derived from short selling the asset $\mathcal{U}_1$ does not suffice to open the desired long position on the other two assets. As mentioned before, we may assume that the investor keeps the same position in the interval $(0, t_1]$, i.e., $\mathcal{A}_1 = \mathcal{A}_0$. The value of the portfolio process at time $t = t_1$ is

$$V(t_1) = V_{\mathcal{A}_1}(t_1) = -400\,\Pi^{\mathcal{U}_1}(t_1) + 200\,\Pi^{\mathcal{U}_2}(t_1) + 100\,\Pi^{\mathcal{U}_3}(t_1).$$

Now suppose that at time $t = t_1$ the investor buys 500 shares of $\mathcal{U}_1$, sells x shares of $\mathcal{U}_2$, and sells all the shares of $\mathcal{U}_3$. Then in the interval $(t_1, t_2]$ the investor has a new portfolio which is given by

$$\mathcal{A}_2 = (100, 200 - x, 0),$$

and so the value of the portfolio process for $t \in (t_1, t_2]$ is given by

$$V(t) = 100\,\Pi^{\mathcal{U}_1}(t) + (200 - x)\,\Pi^{\mathcal{U}_2}(t), \quad t \in (t_1, t_2].$$

The limit of this quantity as $t \to t_1^+$ corresponds to the value of the portfolio “immediately after” the position has been changed at time t_1. Denoting

$$V(t_1^+) = \lim_{t \to t_1^+} V(t)$$

and assuming that the prices are continuous, we have

$$V(t_1^+) = 100\,\Pi^{\mathcal{U}_1}(t_1) + (200 - x)\,\Pi^{\mathcal{U}_2}(t_1).$$

The difference between the value of the two portfolios immediately after and immediately before the transaction is then

$$\begin{aligned} V(t_1^+) - V(t_1) &= 100\,\Pi^{\mathcal{U}_1}(t_1) + (200 - x)\,\Pi^{\mathcal{U}_2}(t_1) \\ &\quad - (-400\,\Pi^{\mathcal{U}_1}(t_1) + 200\,\Pi^{\mathcal{U}_2}(t_1) + 100\,\Pi^{\mathcal{U}_3}(t_1)) \\ &= 500\,\Pi^{\mathcal{U}_1}(t_1) - x\,\Pi^{\mathcal{U}_2}(t_1) - 100\,\Pi^{\mathcal{U}_3}(t_1). \end{aligned}$$

If this difference is positive, then the new portfolio cannot be created from the old one without infusing extra cash. Conversely, if this difference is negative, then the new portfolio is less valuable than the old one, the difference being equivalent to cash withdrawn from the portfolio. Hence for self-financing portfolio processes we must have $V(t_1^+) - V(t_1) = 0$, and similarly $V(t_j^+) - V(t_j) = 0$, for all $j = 1, \ldots, M - 1$. This implies in particular that the number x of shares of the asset $\mathcal{U}_2$ to be sold at time t_1 in a self-financing portfolio must be

$$x = \frac{500\Pi^{\mathcal{U}_1}(t_1) - 100\Pi^{\mathcal{U}_3}(t_1)}{\Pi^{\mathcal{U}_2}(t_1)}.$$

Of course, x will be an integer only in exceptional cases, which means that perfect self-financing strategies in real markets are almost impossible.

If $V(t_j^+) \neq V(t_j)$, i.e., if the portfolio value is discontinuous at time t_j, we say that the portfolio process generates the **cash flow**

$$C(t_j) = -(V(t_j^+) - V(t_j))$$

at time t_j. A positive cash flow corresponds to cash *removed* from the portfolio (causing a decrease of its value), while a negative cash flow corresponds to cash *added* to the portfolio. For instance if at time t_1 the investor sells shares of $\mathcal{U}_1$ and the income is not used to buy shares of another asset, i.e., if it is removed from the portfolio, then $V(t_1^+) < V(t_1)$ and thus $C(t_1) > 0$. The total cash flow generated by the portfolio process in the interval $[0,T]$ is $C_{\mathrm{tot}} = \sum_{j=1}^{M-1} C(t_j)$ and can be negative, positive or zero.

If an asset pays a dividend D at some time $t_* \in (0,T)$, then the portfolio process generates the positive cash flow D at time t_* if the portfolio is long on the asset and the negative cash flow $-D$ if it is short on the asset (because the dividend is due to the original owner of the asset). Constant portfolio positions are self-financing provided the assets pay no dividends.

Portfolios and assets return

Suppose that a portfolio process is opened at time $t = 0$ and closed at time $t = T > 0$, i.e., all positions in the portfolio are liquidated at time T. If the portfolio process is self-financing, then its **return** in the interval $[0,T]$ is given by

$$R(T) = V(T) - V(0), \tag{2}$$

where $V(t)$ denotes the value of the portfolio at time t. If the return is positive, the investor makes a **profit** in the interval $[0,T]$, if it is negative the investor incurs in a **loss**. When $V(0) > 0$ we may also compute the **rate of return** of the portfolio, which is given by

$$R_{\mathrm{rate}}(T) = \frac{V(T) - V(0)}{V(0)}. \tag{3}$$

The total cash flow C generated by a (non-self-financing) portfolio process must be included in the computation of the return of the portfolio in the interval $[0,T]$ according to the formula

$$R(T) = V(T) - V(0) + C. \tag{4}$$

Portfolio returns are commonly "annualized" by dividing the return $R(T)$ by the time T expressed in fraction of years (e.g., $T = 6$ months $= 1/2$ years).

Consider now a portfolio that consists of a long position on one share of the asset $\mathcal{U}$ in the interval $[t, t+h]$ and assume that the asset pays no dividend in this time interval. The annualized rate of return of this portfolio is given by

$$R_h(t) = \frac{\Pi^{\mathcal{U}}(t+h) - \Pi^{\mathcal{U}}(t)}{h\,\Pi^{\mathcal{U}}(t)}$$

and is also called **simply compounded** rate of return of $\mathcal{U}$. In the limit $h \to 0^+$ we obtain the **continuously compounded** (or **instantaneous**) rate of return of the asset:

$$r(t) = \lim_{h\to 0^+} R_h(t) = \frac{1}{\Pi^{\mathcal{U}}(t)} \frac{d\Pi^{\mathcal{U}}(t)}{dt},$$

where we assume that the price of $\mathcal{U}$ is differentiable in time.

Asset returns are often computed using the logarithm of the price rather than the price itself. For instance the quantity

$$\widehat{R}_h(t) = \log \Pi^{\mathcal{U}}(t+h) - \log \Pi^{\mathcal{U}}(t) = \log\left(\frac{\Pi^{\mathcal{U}}(t+h)}{\Pi^{\mathcal{U}}(t)}\right)$$

is called **simply compounded log-return** of the asset $\mathcal{U}$ in the interval $[t, t+h]$. The use of the log-price is convenient in some computations because $\Pi^{\mathcal{U}}(t) > 0$, while $\log \Pi^{\mathcal{U}}(t) \in \mathbb{R}$, i.e., the boundary at zero of the asset price is removed when the log-price is employed. Since $\widehat{R}_h(t)/h$ and $R_h(t)$ have the same limit when $h \to 0^+$, namely

$$\lim_{h\to 0^+} \frac{1}{h}\widehat{R}_h(t) = \lim_{h\to 0^+} \frac{\log \Pi^{\mathcal{U}}(t+h) - \log \Pi^{\mathcal{U}}(t)}{h} = \frac{d \log \Pi^{\mathcal{U}}(t)}{dt} = r(t),$$

then $r(t)$ is also called **instantaneous log-return** of the asset. Note carefully that in general $\widehat{R}_h(t)$, $R_h(t)$ and $r(t)$ are *not* known at time t, because they depend on the future value of the asset $\mathcal{U}$; an exception to this are money market assets discussed later.

Historical volatility

The historical volatility of an asset measures the amplitude of the time fluctuations of the asset price, thereby giving information on its level of uncertainty. It is computed as the standard deviation of the log-returns of the asset based on historical data. More precisely, let $[t_0, t]$ be some interval of time in the past, with t denoting possibly the present time, and let $T = t - t_0 > 0$ be the length of this interval. Let us divide $[t_0, t]$ into n equally long periods, say

$$t_0 < t_1 < t_2 < \cdots < t_n = t, \quad t_i - t_{i-1} = h, \quad \text{for all } i = 1, \dots, n.$$

The set of points $\{t_0, t_1, \dots, t_n\}$ is called a **uniform partition** of the interval $[t_0, t]$. Assume for instance that the asset is a stock. Denote the log-return of the stock price in the interval $[t_{i-1}, t_i]$ as

$$\widehat{R}_i = \log S(t_i) - \log S(t_{i-1}) = \log\left(\frac{S(t_i)}{S(t_{i-1})}\right), \quad i = 1, \dots, n. \tag{5}$$

The average of the log-returns is

$$\widehat{R}(t) = \frac{1}{n}\sum_{i=1}^{n}\widehat{R}_i = \frac{1}{n}\log\left(\frac{S(t)}{S(t_0)}\right). \tag{6}$$

The **T-historical mean of log-return** of the stock is obtained by "annualizing" the average of log-returns, i.e., by dividing $\widehat{R}(t)$ by the length h of the interval in which the log returns are computed:

$$\alpha_T(t) = \frac{1}{nh}\log\left(\frac{S(t)}{S(t_0)}\right) = \frac{1}{T}\log\left(\frac{S(t)}{S(t_0)}\right) \quad (T\text{-historical mean of log-return}). \tag{7}$$

The (corrected) sample variance of the log-returns is

$$\Delta(t) = \frac{1}{n-1}\sum_{i=1}^{n}(\widehat{R}_i - \widehat{R}(t))^2.$$

The **T-historical variance** of the stock is obtained by "annualizing" $\Delta(t)$, i.e.,

$$\sigma_T^2(t) = \frac{1}{h}\frac{1}{n-1}\sum_{i=1}^{n}(\widehat{R}_i - \widehat{R}(t))^2 \quad (T\text{-historical variance}). \tag{8}$$

The square root of the $T-$historical variance is the **T-historical volatility** of the stock:

$$\sigma_T(t) = \frac{1}{\sqrt{h}}\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(\widehat{R}_i - \widehat{R}(t))^2} \quad (T\text{-historical volatility}). \tag{9}$$

Note carefully that the historical volatility of the stock depends on the partition being used to compute it.

Suppose for example that $t - t_0 = T = 20$ days, which is quite common in the applications, and let $t_1, \dots, t_{20}$ be the market closing times at these days. Let $h = 1$ day $= 1/365$ years. Then

$$\sigma_{20}(t) = \sqrt{365}\sqrt{\frac{1}{19}\sum_{i=1}^{n}(\widehat{R}_i - \widehat{R}(t))^2}$$

is called the 20days-historical volatility. Two examples of the curve $t \to \sigma_{20}(t)$ are snown in Figure 1.

Remark 1. The factor $h = 1/252$ is also commonly used in the calculation of market parameters, since there are 252 trading days in one year (markets are closed in the week-end).

Exercise 1 (**Matlab**). *Write the code for a Matlab function that computes the 20days-historical mean of log-return and volatility of a stock.*

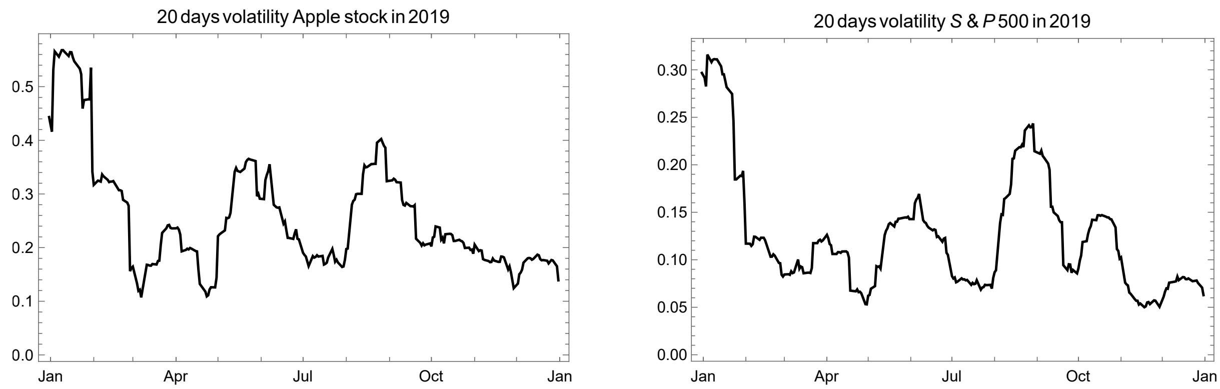

Figure 1: 20-days volatility of the Apple stock and the S&P500 index from January 1st, 2019 until December 31st, 2019.

Assets correlation

Consider again a uniform partition $\{t_0, \ldots, t_n = t\}$ of the past interval $[t_0, t]$ with length $T = t - t_0$. Let $S^{(1)}(t)$, $S^{(2)}(t)$ be the prices of two stocks. Let $\widehat{R}_i^{(1)}, \widehat{R}_i^{(2)}$ be the log-returns of each stock in the interval $[t_{i-1}, t_i]$ and $R^{(1)}, R^{(2)}$ be the averages of log-returns. The **T-historical correlation of log-returns** is computed with the formula

$$\rho_T(t) = \frac{\sum_{i=1}^n (\widehat{R}_i^{(1)} - \widehat{R}^{(1)})(\widehat{R}_i^{(2)} - \widehat{R}^{(2)})}{\sqrt{\sum_{i=1}^n (\widehat{R}_i^{(1)} - \widehat{R}^{(1)})^2 \sum_{i=1}^n (\widehat{R}_i^{(2)} - \widehat{R}^{(2)})^2}}. \tag{10}$$

Denoting by a_1, a_2 the n-dimensional vectors $a_j = (\widehat{R}_1^{(j)} - \widehat{R}^{(j)}, \widehat{R}_2^{(j)} - \widehat{R}^{(j)}, \ldots, \widehat{R}_n^{(j)} - \widehat{R}^{(j)})$, $j = 1, 2$, we can rewrite $\rho_T(t)$ as

$$\rho_T(t) = \frac{a_1 \cdot a_2}{|a_1||a_2|} = \cos\theta,$$

where $\cdot$ denotes the inner product of vectors, $|a_j|$ is the norm of the vector a_j and $\theta \in [0, \pi]$ is the angle between a_1 and a_2. Hence $\rho_T(t) \in [-1, 1]$ and the closer is $\rho_T(t)$ to 1 (resp. -1) the more the stock prices have tendency to move in the same (resp. opposite) direction.

Exercise 2 (?). *Do you see any correlation between the volatility values in Figure 1? If so, how would you interpret this behavior?*

Exercise 3 (**Matlab**). *Write the code for a Matlab function that computes the 20days-historical correlation of two stocks.*

Financial derivatives. Options

A **financial derivative** (or **derivative security**) is an asset whose value depends on the performance of one (or more) other asset(s), which is called the **underlying asset**. There exist several types of financial derivatives, the most common being options, futures, forwards and swaps. Derivatives are available on many different types of assets, including currencies, market indexes, bonds, commodities, etc. In this section we discuss option derivatives on a single asset, which could be for instance a stock.

A **call option** is a contract between two parties: the buyer, or **owner**, of the call and the seller, or **writer**, of the call. The contract gives the owner the right, but *not* the obligation, to buy the underlying asset for a given price, which is fixed at the time when the contract is stipulated, and which is called **strike price** of the call. If the buyer can exercise this right only at some given time T in the future then the call option is called **European**, while if the option can be exercised at any time earlier than or equal to T, then the option is called **American**. The time T is called **maturity time**, or **expiration date** of the call. The writer of the call is obliged to sell the asset to the buyer if the latter decides to exercise the option. If the option to buy in the definition of a call is replaced by the option to sell, then the option is called a **put option**.

In exchange for the option, the buyer must pay a **premium** to the seller (options are not free). Suppose that the option is a European option with strike price K and maturity T. Assume that the underlying asset is a stock with price $S(t)$ at time $t \leq T$ and let Π_0 be the premium paid by the buyer to the seller. In which case is it then convenient for the buyer to exercise the option at maturity? Let us define the **pay-off** of the European call as

$$Y_{\text{call}} = (S(T)-K)_+ := \max(0, S(T)-K) = \begin{cases} 0 & \text{if } S(T) \leq K \\ S(T)-K & \text{if } S(T) > K \end{cases}.$$

Similarly, the pay-off of the European put is defined by

$$Y_{\text{put}} = (K-S(T))_+ = \begin{cases} 0 & \text{if } S(T) \geq K \\ K-S(T) & \text{if } S(T) < K \end{cases}.$$

Clearly, the buyer should exercise the call option at maturity if and only if $Y_{\text{call}} > 0$, as in this case it is cheaper to buy the stock at the strike price rather than at the market price. Similarly the owner of the put should exercise if and only if $Y_{\text{put}} > 0$, as in this case the income generated by selling the stock at the strike price is higher then the income generated by selling it at the market price. Hence the call or put option must be exercised at maturity if and only if the pay-off is positive, in which case the option is said to **expire in the money**. The return for the owner of the option is given by $N(Y_{\text{call}} - \Pi_0)$ in the case of the call and by $N(Y_{\text{put}} - \Pi_0)$ in the case of the put, where N is the number of option contracts in the buyer portfolio. Note carefully that the buyer makes a profit only if the pay-off is greater than the premium. One of the main problems in options pricing theory is to define a reasonable fair value for the price Π_0 of options (and other derivatives).

Let us introduce some further terminology. The European call (resp. put) with strike K is said to be **in the money** at time t if $S(t) > K$ (resp. $S(t) < K$). The call (resp. put) is said to be **out of the money** at time t if $S(t) < K$ (resp. $S(t) > K$). If $S(t) = K$, the (call or put) option is said to be **at the money** at time t. The meaning of this terminology is self-explanatory, see Figure 2.

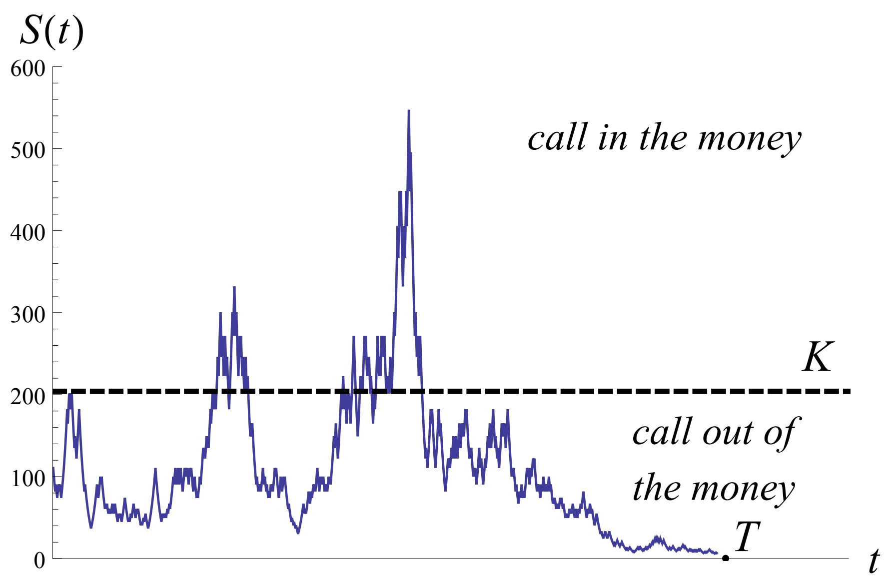

Figure 2: The call option with strike $K = 200$ is in the money in the upper region and out of the money in the lower region. The put option with the same strike is in the money in the lower region and out of the money in the upper region.

The pay-off of the American call exercised at time t is $Y(t) = (S(t) - K)_+$, while for the American put it is given by $Y(t) = (K - S(t))_+$. The quantity $Y(t)$ is also called **intrinsic value** of the American option. In particular, the intrinsic value of an out-of-the-money American option is zero.

Option markets

Option markets are relatively new compared to stock markets. The first one has been established in Chicago in 1974 (the Chicago Board Options Exchange, CBOE). Market options are available on different assets (stocks, debts, indexes, etc.) and with different strikes and maturities. Most commonly, market options are of American style.

Clearly, the deeper in the money is the option, the higher will be its price in the market, while the price of an option deeply out of the money is usually quite low (but never zero!). It is also clear that the buyer of the option is the party holding the long position on the option, since the buyer owns the option and thus hopes for an increase of its value, while the writer is the holder of the short position.

One reason why investors buy call options is to protect a short position on the underlying asset. Suppose for instance that an investor short-sells 100 shares of a stock at time $t = 0$ for the price $S(0)$. A cautious investor will also buy 100 shares of the American call option on the stock with strike $K \approx S(0)$ and maturity $T > 0$. If at some time $t_0 \in (0, T)$ the price of the stock is no lower than $S(0)$ the investor has the option to exercise the call and thus obtain 100 shares of the stock for the price $K \approx S(0)$. In this way the investor will be able to close the short position on the stock with minimal losses. At the same fashion, investors buy put options to protect a long position on the underlying asset. A trading position (particularly a short position) that is not covered by a suitable security is said to be **naked**.

Of course, speculation is also an important factor in option markets. However *the standard theory of options pricing is firmly based on the interpretation of options as derivative securities and does not take speculation into account.*

European, American and Asian derivatives

By far the majority of financial derivatives, including options other than simple calls and puts, are traded OTC. Before discussing a few examples, it is convenient to introduce a precise mathematical definition of European and American derivatives.

Given a function $g : (0, \infty) \to \mathbb{R}$, the **standard** European derivative with pay-off $Y = g(S(T))$ and maturity time $T > 0$ is the contract that pays to its owner the amount Y at time $T > 0$. Here $S(T)$ is the price of the underlying stock at time T, while g is the **pay-off function** of the derivative (e.g., $g(x) = (x - K)_+$ for European call options, while $g(x) = (K - x)_+$ for European put options). Hence, the pay-off of standard European derivatives depends only on the price of the stock at maturity and not on the earlier history of the stock price. An important example of standard European derivative (other than call and put options) is the **digital option**. Denote by $H(x)$ the **Heaviside function**,

$$H(x) = \begin{cases} 1, & \text{for } x > 0 \\ 0, & \text{for } x \leq 0 \end{cases}, \tag{11}$$

and let $K, L > 0$ be constants expressed in units of some currency (e.g., dollars). The standard European derivative with pay-off function $g(x) = LH(x - K)$ is called **cash-settled digital call option** with strike price K and **notional value** L; this derivative pays the amount L if $S(T) > K$, and nothing otherwise. The **physically-settled digital call option** has the pay-off function $g(x) = xH(x - K)$, which means that at maturity the buyer receives either the stock (when $S(T) > K$), or nothing. Digital options are also called **binary** options. Figure **??** shows the graph of the pay-off function for call, put and digital call options with strike $K = 10$. Drawing the graph of the pay-off function of a derivative helps to get a first insight onto its properties.

Exercise 4. *Given $K, \Delta > 0$, consider the standard European derivative with maturity T and pay-off function $g(x) = (x - K + \Delta)_+ - 2(x - K)_+ + (x - K - \Delta)_+$. Draw the graph of g and derive the range of $S(T)$ for which the derivative expires in the money.*

If the pay-off depends on the history of the stock price during the interval $[0, T]$, and not just on $S(T)$, the contract will be called **non-standard** European derivative. An example of non-standard European derivative is the so-called **Asian call option**, the pay-off of which is given by $Y = (\frac{1}{T}\int_0^T S(t)\,dt - K)_+$.

The value at time t of the European derivative with pay-off Y and maturity T will be denoted by $\Pi_Y(t)$ (the expiration date is not included in the notation).

The term "European" signifies that the contract cannot be exercised before maturity T. For a **standard American derivative** the buyer can exercise the contract at any time $t \in (0, T]$ and so doing the buyer will receive the amount $Y(t) = g(S(t))$, where g is the pay-off function of the American derivative. Non-standard American derivatives can be defined similarly to the European ones, but with the further option of earlier exercise. The price at time t of the American derivative with intrinsic value $Y(t)$ and maturity T will be denoted $\widehat{\Pi}_Y(t)$.

Remark 2. The terminology "standard" and "non-standard" derivative is used in this text for easy reference. It is *not* employed in the financial world.

Forward contracts

A **forward contract** with **delivery price** K and maturity (or delivery) time T on an asset $\mathcal{U}$ is a European type financial derivative stipulated by two parties in which one agrees to sell (and possibly deliver) to the other the asset $\mathcal{U}$ at time T in exchange for the cash K. As opposed to options, forward contracts give the same right/obligation to the two parties, as they are both *obliged* to fulfil their part of the agreement at maturity T (buy or sell the asset for the price K). In particular, as there is no privileged position in a forward contract, neither of the two parties has to pay a premium when the contract is stipulated, that is to say, *forward contracts are free*; in fact, the terminology used for forward contracts is "to enter a forward contract" and not "to buy/sell a forward contract". The party who must sell the asset at maturity is said to hold the short position on the forward, while the party who must buy the asset is said to hold the long position, although strictly speaking this terminology refers to the type of position on the underlying asset rather than on the forward contract (which has zero value at all times). Hence the pay-off for a long position in a forward contract on the asset $\mathcal{U}$ is

$$Y_{\text{long}} = (\Pi^{\mathcal{U}}(T) - K),$$

while for the holder of the short position the pay-off is

$$Y_{\text{short}} = (K - \Pi^{\mathcal{U}}(T)).$$

Forward contracts are traded OTC and most commonly on commodities or market indexes, such as currency exchange rates, interest rates and volatilities. In the case that the underlying asset is an index, forward contracts are also called **swaps** (e.g., currency swaps, interest rate swaps, volatility swaps, etc.).

One purpose of forward contracts is to share risks. Irrespective of the movement of the underlying asset in the market, its price at time T for the holders of the forward contract will be K. The delivery price agreed by the two parties in a forward contract is also called the **forward price** of the asset. More precisely, the T-forward price $\mathrm{For}_{\mathcal{U}}(t,T)$ of an asset $\mathcal{U}$ at time $t < T$ is the delivery price of a forward contract on $\mathcal{U}$ stipulated at time t and with maturity T, while the current, actual price $\Pi^{\mathcal{U}}(t)$ of the asset is called the **spot price**.

Futures contracts

Futures are standardized forward contracts listed in official exchange markets, called **futures market**, which include for instance the Chicago Mercantile Exchange (CME), the New York Mercantile Exchange (NYMEX), the Chicago Board of Trade (CBOT) and the International Exchange Group (ICE). Unlike forward contracts, all futures contracts in a futures market are subject to the same regulation.

The T-**future price** $\mathrm{Fut}_{\mathcal{U}}(t,T)$ of the asset $\mathcal{U}$ at time $t \leq T$ is defined as the delivery price at time $t \leq T$ in the futures contract with maturity T on the asset $\mathcal{U}$. Holding a position in a futures contract in the futures market consists in the agreement to receive as a cash flow the change in the future price of the underlying asset during the time in which the position is held. The cash flow may be positive or negative. In a long position the cash flow is positive when the future price goes up and it is negative when the future price goes down. The cash flow is distributed in time through the so called **margin account**. For example, assume that at $t = 0$ an investor opens a long position in a futures contract expiring at time T. At the same time, the investor needs to open a margin account which contains a certain amount of cash (usually, 10 % of the current value of the T-future price for each contract opened). At $t = 1$ day, the amount $\mathrm{Fut}_{\mathcal{U}}(1,T) - \mathrm{Fut}(0,T)$ will be added to the account, if it positive, or withdrawn, if it is negative. The position can be closed at any time $t < T$ (multiple of days), in which case the total amount of cash flown in the margin account is

$$
\begin{aligned}
&(\mathrm{Fut}_{\mathcal{U}}(t,T) - \mathrm{Fut}_{\mathcal{U}}(t-1,T)) + (\mathrm{Fut}_{\mathcal{U}}(t-1,T) - \mathrm{Fut}_{\mathcal{U}}(t-2,T)) + \\
&\quad \dots + (\mathrm{Fut}_{\mathcal{U}}(1,T) - \mathrm{Fut}_{\mathcal{U}}(0,T)) = (\mathrm{Fut}_{\mathcal{U}}(t,T) - \mathrm{Fut}_{\mathcal{U}}(0,T)).
\end{aligned}
$$

If the long position is held up to the time of maturity, then the investor should buy the underlying asset. However futures contracts are often **cash settled** and not **physically settled**, which means that the delivery of the underlying asset does not occur, and the equivalent value in cash is paid instead.

An **option on futures** with maturity $T > 0$ and strike K is a contract that gives to the owner the right to enter at time T in a futures contract (expiring at time $S > T$) at the future price K. In the case of a call (resp. put) option, the owner has the right to take a long (resp. short) position on the futures contract and thus the pay-off will be $(\mathrm{Fut}_{\mathcal{U}}(T,S) - K)_+$ (resp. $(K - \mathrm{Fut}_{\mathcal{U}}(T,S))_+$). If the option on futures expires in the money, the owner can decide to keep open the position on the futures contract or to close it immediately, thereby cashing the pay-off of the option. Options on futures are example of **second derivatives**, i.e., financial derivatives whose underlying asset is another derivative.

Bonds

The **zero-coupon bond** (ZCB) with **face** (or **nominal**) **value** K and maturity $T > 0$ is the contract that promises to pay to its owner the amount K at time T in the future. Without loss of generality it will be assumed from now on that $K = 1$, as owning one share of the ZCB with face value K is clearly equivalent to own K shares of the ZCB with face value 1. ZCB's (and the related coupon bonds described below) are first issued in the so-called **primary market** by national governments and private companies as a way to borrow money and fund their activities; starting from the following market day, the ZCB's become tradable assets in the **secondary** market and thus their price changes in time. Let $B(t,T)$ denote the value at time t of the ZCB with face value 1 and expiring at time T. If the issuer of the ZCB announces at time $t_0 < T$ that it is unable to comply with the payment of the face value at maturity, then the ZCB becomes worthless, i.e., $B(t,T) \equiv 0$ for $t \in [t_0, T]$ and the issuer of the ZCB is said to be in **default**. Suppose that the issuer of the ZCB bears no risk of default in the interval $[t,T]$. The investors who own shares of the ZCB at maturity T will then receive at time T the promised face value, multiplied by the number of shares owned, from the original issuer of the ZCB. The return per share of this investment is $R(t) = 1 - B(t,T)$, where t is the time at which the investor bought the ZCB. Under normal market conditions, $B(t,T) < 1$, for $t < T$, i.e., the investor pays less than 1 today to receive 1 in the future, and thus $R(t) > 0$. However exceptions are possible; for instance national bonds in Sweden with maturity shorter than 10 years yield currently (2020) a negative return.

Bonds with long maturity typically pay coupons in addition to the face value. Let $0 < t_1 < t_2 < \cdots < t_M = T$ be a partition of the interval $[0,T]$. A **coupon bond** with maturity T, face value 1 and coupons $c_1, c_2, \dots, c_M \in (0,1)$ is a contract that promises to pay the amount c_k at time t_k and the amount $1 + c_M$ at maturity $T = t_M$. Most commonly the coupons are all equal, i.e., $c_1 = c_2 = \cdots = c_M$, and paid annually (or semi-annually). The maturity of coupon bonds can reach up to 30 or more years.

Money market

The **money market** is a component of the debt market consisting of **short term loans**, i.e., loan contracts with maturity between one day and one year. Examples of money market assets are **treasury bills**, i.e., ZCB's with short maturity (less than 1 year), commercial papers, certificates of deposit, saving accounts and repurchase agreements (**repo**). In contrast to stock and option markets, money markets are typically accessible only by financial institutions and not by private investors.

The value at time t of a generic asset in the money market will be denoted by $B(t)$. The difference $B(t_2) - B(t_1)$, $t_1 < t_2$, determines the **interest rate** of the asset in the interval $[t_1, t_2]$. In particular, let $\{t_0 = 0, t_1, \dots, t_N = t\}$ be a uniform partition of the interval $[0,t]$ with size $h = t_i - t_{i-1}$. The money market asset is said to have **simply compounded** interest rate $R_h(s)$ in the time period $[s, s+h]$, where $s \in \{t_0, \dots, t_{N-1}\}$, if the value of the asset satisfies

$$B(s+h) = B(s)(1 + R_h(s)h), \quad s \in \{t_0, \dots, t_{N-1}\}. \tag{12}$$

Inverting (12) we have

$$R_h(s) = \frac{B(s+h) - B(s)}{hB(s)}, \tag{13}$$

i.e., $R_h(s)$ is the annualized rate of return of the asset in the interval $[s, s+h]$. Note carefully that $R_h(s)$ *is known at time* s (as opposed for instance to the return of stocks in the interval $[s, s+h]$, which is not known at time s). Iterating (12) the value at time $t = t_N$ of the risk-free asset can be expressed in terms of the value at time $t = 0$ by the formula

$$\begin{aligned} B(t) &= B(t_{N-1})(1 + R_h(t_{N-1})h) = B(t_{N-2})(1 + R_h(t_{N-2})h)(1 + R_h(t_{N-1})h) \\ &= \dots = B(0) \prod_{i=0}^{N-1} (1 + R_h(t_i)h). \end{aligned} \tag{14}$$

Example. Suppose that at time $t_0 = 0$ an investor is borrowing the quantity $B(0) =$ 1000000 Kr for one year with 3-months compounded interest rate, i.e., $h = 1/4$. Suppose $R_{1/4}(t_0) = 0.03$ in the first quarter, $R_{1/4}(t_1) = 0.02$ in the second quarter, $R_{1/4}(t_2) = 0.01$ in the third quarter and $R_{1/4}(t_3) = 0.04$ in the last quarter. Here $t_0 = 0$, $t_1 = 1/4$, $t_2 = 1/2$, $t_3 = 3/4$. The debt of the investor at time $t_4 = 1$ year is

$$B(t_4) = B(t_0)(1 + \frac{1}{4}R_{1/4}(t_0))(1 + \frac{1}{4}R_{1/4}(t_1))(1 + \frac{1}{4}R_{1/4}(t_2))(1 + \frac{1}{4}R_{1/4}(t_3)) \approx 1025220 \text{ Kr.}$$

If the investor borrows instead at the yearly compounded rate $R_1(t_0) = 0.03$ (i.e., $h = 1$), the debt after 1 year is $B(t_4) = B(t_0)(1 + R_1(t_0)) = 1030000$ Kr. Notice that at time $t = t_0$ the investor knows $R_{1/4}(t_0)$ and $R_1(t_0)$ but does not know the values of $R_{1/4}(t_1), R_{1/4}(t_2), R_{1/4}(t_3)$ and thus cannot anticipate whether it is more convenient to borrow at variable or constant interest rate. Investors may use financial instruments such as **interest rate swaps** or **interest rate caps/floors** to hedge against the risk derived from the fluctuations of interest rates in the market.

Letting $h \to 0$ in (13) we obtain the **continuously compounded** interest rate (or **short rate**) $r(t)$ of the money market asset, namely

$$R_h(s) \to r(s) = \frac{B'(s)}{B(s)} = \frac{d}{ds} \log B(s), \quad \text{as } h \to 0. \tag{15}$$

Thus $r(t)$ is the interest rate to borrow at time t for an "infinitesimal" interval of time, which in the real world corresponds to overnight loans. Integrating (15) on $[t, t+h]$ we find

$$B(t+h) = B(t)e^{\int_t^{t+h} r(s)\,ds}, \tag{16}$$

which is the continuum analog of (12). Integrating (15) in the time interval $[0, t]$ we obtain the continuum analog of (14), namely

$$B(t) = B(0) \exp\left(\int_0^t r(s)\,ds\right). \tag{17}$$

Frictionless markets

Market models in financial mathematics are based on a number of simplifying assumptions which deviate, sometime substantially, from the behavior of real markets. Among these simplifying assumptions we impose that

1. There is no bid/ask spread
2. There are no transaction costs and trades occur instantaneously
3. An investor can trade any fraction of shares
4. When a stock pays a dividend, the ex-dividend date and the payment date are the same and the stock price at this date drops by the exact same amount paid by the dividend

As seen in the previous sections real markets do not satisfy exactly these assumptions, although in some case they do it with reasonable approximation. For instance, if the investor is a professional agent managing large portfolios then the above assumptions reflect reality quite well. However they work very badly for private investors and for small portfolios. The validity of these assumptions is summarized by saying that the market has **no friction**. The idea is that, when the above assumptions hold, trading proceeds "smoothly without resistance".

In a frictionless market the portfolio process of an agent who is investing on N assets during the time interval $[0, T]$ may be defined as a function

$$\mathcal{A} : [0, T] \to \mathbb{R}^N, \quad \mathcal{A}(t) = (a_1(t), \dots, a_N(t)),$$

i.e., by assumptions 2 and 3, the number of shares $a_i(t)$ of each single asset at time t is now allowed to be any real number and to change at any arbitrary time in the interval $[0, T]$; of course, in real market applications $a_1(t), \dots, a_N(t)$ must be rounded to integer numbers. Portfolio processes can be added using the linear structure in $\mathbb{R}^N$, namely if $\mathcal{B} = (b_1(t), \dots, b_N(t))$, and $\alpha, \beta \in \mathbb{R}$, then $\alpha\mathcal{A} + \beta\mathcal{B}$ is the portfolio process

$$\alpha\mathcal{A} + \beta\mathcal{B} = (\alpha a_1(t) + \beta b_1(t), \dots, \alpha a_N(t) + \beta b_N(t)).$$

The value at time t of the portfolio process $\mathcal{A}$ is

$$V_{\mathcal{A}}(t) = \sum_{i=1}^{N} a_i(t)\Pi^{\mathcal{U}_i}(t),$$

and clearly

$$V_{\alpha\mathcal{A}}(t) + V_{\beta\mathcal{B}}(t) = V_{\alpha\mathcal{A}+\beta\mathcal{B}}(t).$$

Moreover, thanks to assumption 3, perfect self-financing portfolio processes in frictionless markets always exist.

By assumption 1, any offer to buy/sell an asset is matched by an offer to sell/buy the asset. Of course this assumption is only reasonable when the price of the asset is *fair*. What exactly means that asset prices are fair is explained at the end of this text.

Rational investor principle

The purpose of this and the following two sectios is to introduce a number of basic fundamental principles in financial mathematics. The following notation will be used. $S(t)$ denotes the price at time $t > 0$ of a given stock, $C(t, S(t), K, T)$ denotes the price at time $t \in [0, T]$ of the European call option on the stock with strike $K > 0$ and maturity $T > 0$. The price of the European put option with the same parameters will be denoted by $P(t, S(t), K, T)$; finally $\widehat{C}(t, S(t), K, T)$ and $\widehat{P}(t, S(t), K, T)$ denote the values of the corresponding American call and put option.

Probably the most self-evident of all financial principles is the **rational investor principle**:

Investors prefer more to less and do not undertake trading strategies which result in a sure loss.

This principle has a number of straightforward consequences. For example, an investor will never exercise an option which is out of the money, while an option that expires in the money is always exercised. Moreover the price of stocks and options is always non-negative[1]. The following is a short list of simple properties of financial derivatives implied by the rational investor principle, and whose justification is left to the reader:

(i) The price of a financial derivative tends to its pay-off as maturity is approached. In particular, for European call/put options,

$$C(t, S(t), K, T) \to (S(T) - K)_+, \quad P(t, S(t), K, T) \to (K - S(T))_+,$$

as $t \to T^-$ and similarly for American options, while for the ZCB with maturity T and face value 1 there holds

$$\lim_{t \to T^-} B(t, T) = 1.$$

(ii) An American derivative is at least as valuable as its European counterpart. In particular, for call/put options,

$$\widehat{C}(t, S(t), K, T) \geq C(t, S(t), K, T), \quad \widehat{P}(t, S(t), K, T) \geq P(t, S(t), K, T).$$

(iii) The price of an American derivative is always larger or equal to its intrinsic value. In particular, for American call/put options,

$$\widehat{C}(t, S(t), K, T) \geq (S(t) - K)_+, \quad \widehat{P}(t, S(t), K, T) \geq (K - S(t))_+.$$

[1] The limiting case $S(t) = 0$ (zero stock price) models the **default** of the stock.

(iv) The price of European and American call options at time t is no larger than the price of the underlying asset at time t, i.e.,

$$C(t, S(t), K, T) \leq S(t) \quad \text{and} \quad \widehat{C}(t, S(t), K, T) \leq S(t).$$

Any reasonable mathematical model for the price of options must be consistent with the properties (i)-(iv) in the previous exercise.

By (i) and integrating (15) in the time interval $[t, T]$, the value at time t of the ZCB can be written as

$$B(t, T) = e^{-\int_t^T r(s)\, ds}. \tag{18}$$

Existence of risk-free assets

An asset $\mathcal{U}$ is said to be **risk-free** in the interval $[t, T]$ if the value of $\mathcal{U}$ at time T is known at time t. For instance, a ZCB with face value 1 and maturity $T > 0$ issued at time $t = 0$ satisfies $B(T, T) = 1$ and thus it is risk-free in any interval $[t, T]$, $0 \leq t < T$. Similarly, the rate of return of money market assets in a sufficiently short period of time $[t, t + h]$ is known at time t and so these assets are risk-free in the interval $[t, t + h]$. Note however that ZCB's and money market assets can be considered risk-free *only if the borrower party bears no risk of default.* In the real world it is impossible to exclude with certainty the default of a financial institution, but this event can be sometimes considered very unlikely within a reasonable short time in the future. For instance, while there is no general consensus on this, many investors believe that the US treasure bills are actual risk-free assets. In this text we make the following assumption.

There exist risk-free assets in the money market.

In a frictionless market the interest rate of all risk-free assets in the money market must necessarily be the same, otherwise one would generate a profit by borrowing at the lower rate and lending at the higher rate (this is an example of arbitrage opportunity, see Definition 1 below). The (hypothetical) common short rate of all risk-free assets in the money market will be referred to as the **risk-free rate**. Which market parameter represents a realistic estimate for the value of the risk-free rate is an important and constantly debated issue in finance. A popular choice is the yield of domestic treasure bills. Another frequent choice is to identify the risk-free rate with an **interbank offered rate**, such as LIBOR, or EURIBOR, etc., i.e., the average interest rate at which banks in a given geographical zone lend money to one another. An alternative approach is to interpret the risk-free rate as an **implied parameter**, i.e., a parameter whose value is determined by calibrating a mathematical model for the market dynamics.

Arbitrage-free principle

The next principle is based on the fundamental concept of arbitrage portfolio process, which is defined as follows.

Definition 1. *Let t be the present time and $T > t$. A portfolio process $\mathcal{A}$ is called an* **arbitrage** *in the interval $[t, T]$ if*

(a) $V_{\mathcal{A}}(t) = 0$*;*

(b) It is known at time t that the return of $\mathcal{A}$ is positive in the interval $[t, T]$.

Hence an arbitrage portfolio is an investment strategy that requires no initial wealth and which ensures a positive profit without taking any risk. For example, suppose that at time $t = 0$ an investor sells one share the American call option with strike K and maturity T_1 and buys one share of the American call on the same stock with the same strike but with maturity $T_2 > T_1$. Suppose that the price of the latter option is lower than the price of the former, i.e., $\widehat{C}_2 := \widehat{C}(0, S(0), K, T_2) < \widehat{C}(0, S(0), K, T_1) := \widehat{C}_1$. The investor will then have the cash $\widehat{C}_1 - \widehat{C}_2$ available to buy shares of a risk-free asset in the money market. This (constant) portfolio is clearly an arbitrage in any interval $[0, T] \subseteq [0, T_1]$. In fact it has zero initial value and if the owner of the option with maturity T_1 decides to exercise at some time $T \leq T_1$, the investor can pay-off the buyer by exercising the option in the portfolio; the remaining value in the portfolio at time T would equal the (positive) value of the risk-free asset.

Despite appearing "too good to be true", arbitrage opportunities do actually exist in real markets, but last only for a very short time, as they are quickly exploited and "traded away" by investors[2]. In this text arbitrage opportunities are neglected altogether by imposing the **arbitrage-free principle**:

Asset prices are such that no arbitrage can be found in the market.

Asset prices in an arbitrage-free market are also said to be **fair** (or **arbitrage-free**).

References

[1] S. Calogero: *A first course on options pricing theory.* Lecture notes for the course "Options and Mathematics" at Chalmers. Available on the course homepage.

[2] Investors who try to make profits by exploiting arbitrage opportunities in the market are called **arbitragers**.