Financial derivatives and PDE's
Lecture 1

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1 Probability space

Notation:

Ω non-empty set,

2^Ω power set of Ω (set of all subsets of Ω),

$\mathcal{F} \subseteq 2^\Omega$ σ-algebra of subsets of Ω.

A **probability space** is a triple $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P} : \mathcal{F} \to [0,1]$ is a **probability measure**, i.e., a measure such that $\mathbb{P}(\Omega) = 1$.

Ω is called **sample space**.

Two probability measures $\mathbb{P}, \widetilde{\mathbb{P}} : \mathcal{F} \to [0,1]$ are said to be **equivalent** if $\mathbb{P}(A) = 0 \Leftrightarrow \widetilde{\mathbb{P}}(A) = 0$.

Case 1: Ω is countable

In this case $\Omega = \{\omega_n\}_{n\in\mathbb{N}}$ and $\mathcal{F} = 2^\Omega$. Let $\{p_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers $p_n \in (0,1)$ such that $\sum_n p_n = 1$. We set

$$\mathbb{P}(A) = \sum_{n:\omega_n\in A} p_n, \quad A \in 2^\Omega, \quad \mathbb{P}(\emptyset) = 0.$$

The empty set is the only set with zero probability. In particular, all probability measures in a countable sample space are equivalent.

Case 2: Ω is uncountable

In this case we pick $\mathcal{F} \subset 2^\Omega$ generated by a family $\mathcal{O} \subset 2^\Omega$ of subsets of Ω, i.e., $\mathcal{F} = \mathcal{F}_\mathcal{O}$, where

$$\mathcal{F}_\mathcal{O} = \bigcap\{\sigma\text{-algebras } \mathcal{G} : \mathcal{O} \subseteq \mathcal{G}\}$$

is the smallest σ-algebra containing $\mathcal{O}$.

Example: $\mathcal{O} = \{\text{open real intervals}\}$, then $\mathcal{F}_\mathcal{O} \equiv \mathcal{B}(\mathbb{R})$ (**Borel σ-algebra**). Given $f : \mathbb{R} \to [0,\infty)$ measurable such that

$$\int_\mathbb{R} f(x)\, dx = 1 \quad \text{(Lebesgue integral)},$$

then $\mathbb{P}(A) = \int_A f(x)\, dx$ defines a probability measure on $\mathcal{B}(\mathbb{R})$.

Given $A, B \in \mathcal{F} : \mathbb{P}(B) > 0$, the quantity $\mathbb{P}(A|B) = \mathbb{P}(A \cap B)/\mathbb{P}(B)$ is called probability of the event A **conditional** to the event B.

If $\mathbb{P}(A|B) = \mathbb{P}(A)$, i.e., $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$, then we say that A, B are **independent**.

Two sub-σ-algebras $\mathcal{H}, \mathcal{G} \subset \mathcal{F}$ are independent if every event in $\mathcal{H}$ is independent of every event in $\mathcal{G}$.

A **filtration** is a one parameter family $\{\mathcal{F}(t)\}_{t\geq 0}$ of σ-algebras such that $\mathcal{F}(t) \subseteq \mathcal{F}$ for all $t \geq 0$ and $\mathcal{F}(s) \subseteq \mathcal{F}(t)$ for all $s \leq t$.

A quadruple $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\geq 0}, \mathbb{P})$ is called a **filtered probability space**.

2 Random Variables

$X : \Omega \to \mathbb{R}$ is a **random variable** if $\{X \in U\} \in \mathcal{F}$ for all $U \in \mathcal{B}(\mathbb{R})$, where $\{X \in U\}$ denotes the pre-image of the Borel set U.

Sets of the form $\{X \in U\}$ are events whose occurrence can be inferred by knowing the value of X, hence they form the so-called **information carried by** X, or σ**-algebra generated by** X:

$$\sigma(X) = \{A \in \mathcal{F} : A = \{X \in U\}, \text{ for some } U \in \mathcal{B}(\mathbb{R})\}.$$

If $X(\omega) = c$ for all $\omega \in \Omega$, we call X a **deterministic constant**. Clearly $\sigma(c) = \{\emptyset, \Omega\}$.

If X, Y are random variables we let $\sigma(X, Y) = \mathcal{F}_{\mathcal{O}}$, where $\mathcal{O} = \sigma(X) \cup \sigma(Y)$.

If $\Omega = \mathbb{R}$ and $\mathcal{F} = \mathcal{B}(\mathbb{R})$ then the random variable is denoted by a small Latin letter (e.g., $f : \mathbb{R} \to \mathbb{R}$) and called **measurable function**.

If X is a random variable and f is a measurable function, then $Y = f(X)$ is a random variable. In this case $\sigma(Y) \subseteq \sigma(X)$ (and thus $\sigma(X, Y) = \sigma(X)$). The opposite is also true: if $\sigma(Y) \subseteq \sigma(X)$ then there exists a measurable function f such that $Y = f(X)$.

Two random variables X, Y are **independent** if $\sigma(X)$, $\sigma(Y)$ are independent σ-algebras. In this case $\sigma(X) \cap \sigma(Y)$ consists of **trivial events** only, i.e., events with probability zero or one.

If $A \in \mathcal{F}$ we denote $\mathbb{I}_A$ the random variable

$$\mathbb{I}_A(\omega) = \begin{cases} 1, & \omega \in A, \\ 0, & \omega \in A^c. \end{cases}$$

Clearly $\sigma(\mathbb{I}_A) = \{\emptyset, \Omega, A, A^c\}$.

If $\{A_k\}_{k=1,\dots,N} \subset \mathcal{F}$ is a **finite partition** of Ω (i.e., disjoint sets whose union equals Ω) and $a_1, \dots, a_N$ are distinct real numbers, the random variable

$$X = \sum_{k=1}^{N} a_k \mathbb{I}_{A_k}$$

is called a **simple random variable**. In this case $\sigma(X)$ consists of all the sets which can be written as union of the events in the partition (plus the empty set).

3 Distribution functions

The **(cumulative) distribution function** of the random variable $X : \Omega \to \mathbb{R}$ is the non-negative function $F_X : \mathbb{R} \to [0, 1]$ given by $F_X(x) = \mathbb{P}(X \leq x)$.

Two random variables X, Y are said to be **identically distributed** if $F_X = F_Y$.

Properties: F_X is (1) right-continuous, (2) non-decreasing, (3) $\lim_{x\to+\infty} F_X(x) = 1$ and $\lim_{x\to-\infty} F_X(x) = 0$

A random variable $X : \Omega \to \mathbb{R}$ is said to admit the **probability density function (pdf)** $f_X : \mathbb{R} \to [0, \infty)$ if f_X is integrable on $\mathbb{R}$ and

$$F_X(x) = \int_{-\infty}^{x} f_X(y)\, dy. \tag{1}$$

Note that if f_X is the pdf of a random variable, then necessarily

$$\int_{\mathbb{R}} f_X(x)\, dx = \lim_{x\to\infty} F_X(x) = 1.$$

The density f_X exists in particular when F_X is differentiable and in this case we have

$$f_X = \frac{dF_X}{dx}.$$

Examples.

1. A random variable $X : \Omega \to \mathbb{R}$ is said to be a **normal** (or **normally distributed**) random variable if it admits the density

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\dfrac{(x-m)^2}{2\sigma^2}},$$

for some $m \in \mathbb{R}$ and $\sigma > 0$, which are called respectively the **expectation** (or **mean**) and the **deviation** of the normal random variable X, while σ^2 is called the **variance** of X. We denote by $\mathcal{N}(m, \sigma^2)$ the set of all normal random variables with expectation m and variance σ^2. If $m = 0$ and $\sigma^2 = 1$, $X \in \mathcal{N}(0, 1)$ is said to be a **standard** normal variable. The density function of standard normal random variables is denoted by ϕ, while their distribution is denoted by Φ, i.e.,

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}}\, dy.$$

2. A random variable $X : \Omega \to \mathbb{R}$ is said to be **non-central chi-squared distributed** with **degree** $\delta > 0$ and **non-centrality parameter** $\beta > 0$ if it admits the density

$$f_X(x) = \frac{1}{2} e^{-\frac{x+\beta}{2}} \left(\frac{x}{\beta}\right)^{\frac{\delta}{4}-\frac{1}{2}} I_{\delta/2-1}(\sqrt{\beta x}) \mathbb{I}_{x>0}, \tag{2}$$

where $I_\nu(y)$ denotes the modified Bessel function of the first kind. We denote by $\chi^2(\delta, \beta)$ the random variables with density (2).

The **joint (cumulative) distribution** $F_{X,Y} : \mathbb{R}^2 \to [0,1]$ of two random variables $X, Y : \Omega \to \mathbb{R}$ is defined as

$$F_{X,Y}(x,y) = \mathbb{P}(X \leq x, Y \leq y).$$

The random variables X, Y are said to admit the **joint (probability) density function** $f_{X,Y} : \mathbb{R}^2 \to [0,\infty)$ if $f_{X,Y}$ is integrable in $\mathbb{R}^2$ and

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(\eta, \xi)\, d\eta\, d\xi.$$

Note the formal identities

$$f_{X,Y} = \frac{\partial^2 F_{X,Y}}{\partial x\, \partial y}, \quad \int_{\mathbb{R}^2} f_{X,Y}(x,y)\, dx\, dy = 1.$$

As an example of joint pdf, let $m = (m_1, m_2) \in \mathbb{R}^2$ and $C = (C_{ij})_{i,j=1,2}$ be a 2×2 positive definite, symmetric matrix. Two random variables $X, Y : \Omega \to \mathbb{R}$ are said to be **jointly normally distributed** with **mean** m and **covariance matrix** C if they admit the joint density

$$f_{X,Y}(x,y) = \frac{1}{\sqrt{(2\pi)^2 \det C}} \exp\left[-\frac{1}{2}(z-m)\cdot C^{-1} \cdot (z-m)^T\right], \tag{3}$$

where $z = (x, y)$, " $\cdot$ " denotes the row by column product, C^{-1} is the inverse matrix of C and v^T is the transpose of the vector v.