Financial derivatives and PDE's
Lecture 4

Simone Calogero

January 22^{th}, 2021

1 Ito's integral

Given a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\geq 0}, \mathbb{P})$ with a Brownian motion $\{W(t)\}_{t\geq 0}$ such that $\{\mathcal{F}(t)\}_{t\geq 0}$ is non-anticipating, we denote by $\mathcal{C}^0[\mathcal{F}(t)]$ the set of stochastic processes adapted to $\{\mathcal{F}(t)\}_{t\geq 0}$ and with a.s. continuous paths.

For instance, for $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ continuous function, the processes

$$\{f(t, W(t))\}_{t\geq 0}, \quad \{\int_0^t f(s, W(s))\, ds\}_{t\geq 0}$$

belong to $\mathcal{C}^0[\mathcal{F}(t)]$. Moreover if $\{X(t)\}_{t\geq 0} \in \mathcal{C}^0[\mathcal{F}(t)]$, then $\{I(t)\}_{t\geq 0} \in \mathcal{C}^0[\mathcal{F}(t)]$, where $I(t)$ denotes the **Itô integral**

$$I(t) = \int_0^t X(s)\, dW(s), \tag{1}$$

i.e., the integral of the process $\{X(t)\}_{t\geq 0}$ along the paths of the Brownian motion.

If $\{X(t)\}_{t\geq 0}$ has differentiable paths then $I(t)$ is defined as

$$I(t) = -\int_0^t X'(s)W(s)\, ds + X(t)W(t),$$

which is equivalent to "integrating by parts" in (1).

When $\{X(t)\}_{t\geq 0}$ is a general process in $\mathcal{C}^0[\mathcal{F}(t)]$, $I(t)$ is defined as the **limit in probability** of the sum

$$\sum_{j=0}^{m(n)-1} X(t_j^{(n)},\omega)(W(t_{j+1}^{(n)},\omega) - W(t_j^{(n)},\omega))$$

along a sequence a partitions $\Pi_n = \{t_0 = 0, t_1^{(n)}, \dots t_{m(n)} = t\}$ such that $\|\Pi_n\| \to 0$ as $n \to \infty$.

It can be shown that when $\{X(t)\}_{t\geq 0} \in \mathcal{C}^0[\mathcal{F}(t)]$ this limit is independent of the sequence of partitions used to definite it.

We shall not need the precise general definition of Itô integral, but only the fact that it satisfies the properties in the following theorem. Denote

$$\mathbb{L}^2[\mathcal{F}(t)] = \{\{X(t)\}_{t\geq 0} \in \mathcal{C}^0[\mathcal{F}(t)] : \mathbb{E}[\int_0^T X^2(t)\,dt] < \infty, \text{ for all } T > 0\},$$

Theorem 1. *Let $\{X(t)\}_{t\geq 0} \in \mathcal{C}^0[\mathcal{F}(t)]$. Then the Itô integral*

$$I(t) = \int_0^t X(s)dW(s) \tag{2}$$

satisfies the following properties for all $t \geq 0$.

(0) $\{I(t)\}_{t\geq 0} \in \mathcal{C}^0[\mathcal{F}(t)]$. *If* $\{X(t)\}_{t\geq 0} \in \mathbb{L}^2[\mathcal{F}(t)]$, *then* $\{I(t)\}_{t\geq 0} \in \mathbb{L}^2[\mathcal{F}(t)]$.

(i) **Linearity:** *For all stochastic processes* $\{X_1(t)\}_{t\geq 0}, \{X_2(t)\}_{t\geq 0} \in \mathcal{C}^0[\mathcal{F}(t)]$ *and real constants* $c_1, c_2 \in \mathbb{R}$ *there holds*

$$\int_0^t (c_1X_1(s) + c_2X_2(s))dW(s) = c_1\int_0^t X_1(s)dW(s) + c_2\int_0^t X_2(s)dW(s).$$

(ii) **Martingale property:** *If* $\{X(t)\}_{t\geq 0} \in \mathbb{L}^2[\mathcal{F}(t)]$, *then the stochastic process* $\{I(t)\}_{t\geq 0}$ *is a martingale in the filtration* $\{\mathcal{F}(t)\}_{t\geq 0}$. *In particular,* $\mathbb{E}[I(t)] = \mathbb{E}[I(0)] = 0$, *for all* $t \geq 0$.

(iii) **Quadratic variation:** *For all $T > 0$, the quadratic variation of the stochastic process $\{I(t)\}_{t\geq 0}$ on the interval $[0,T]$ is independent of the sequence of partitions along which it is computed and it is given by*

$$[I,I](T) = \int_0^T X^2(s)\,ds, \quad \text{i.e., } dI(t)dI(t) = X^2(t)\,dt. \tag{3}$$

(iv) **Itô's isometry:** *If* $\{X(t)\}_{t\geq 0} \in \mathbb{L}^2[\mathcal{F}(t)]$*, then* $\mathrm{Var}[I(t)] = \mathbb{E}[I^2(t)] = \mathbb{E}[\int_0^t X^2(s)\,ds]$*, for all* $t \geq 0$*.*

*(v) (***Martingale representation theorem***). If* $\{X(t)\}_{t\geq 0} \in \mathbb{L}^2[\mathcal{F}_W(t)]$ *is a martingale relative to* $\{\mathcal{F}_W(t)\}_{t\geq 0}$*, then there exists a stochastic process* $\{\Gamma(t)\}_{t\geq 0} \in \mathbb{L}^2[\mathcal{F}_W(t)]$ *such that*

$$M(t) = M(0) + \int_0^t \Gamma(s)dW(s).$$

Important: Note carefully that the filtration used in the martingale representation theorem must be the one generated by the Brownian motion.

We denote the Itô integral of stochastic processes by using the **stochastic differential notation**, namely

$$dI(t) = X(t)dW(t).$$

Employing this notation, the proof of property (iii) in Theorem 1 is straightforward:

$$dI(t)dI(t) = X(t)^2 dW(t)dW(t) = X^2(t)dt.$$

Given $\{\alpha(t)\}_{t\geq 0}, \{\sigma(t)\}_{t\geq 0} \in \mathcal{C}^0[\mathcal{F}(t)]$, the stochastic process $\{X(t)\}_{t\geq 0} \in \mathcal{C}^0[\mathcal{F}(t)]$ given by

$$X(t) = X(0) + \int_0^t \sigma(s)dW(s) + \int_0^t \alpha(s)\,ds, \quad t \geq 0 \tag{4}$$

is called **diffusion process** with **rate of quadratic variation** $\{\sigma^2(t)\}_{t\geq 0}$ and **drift** $\{\alpha(t)\}_{t\geq 0}$.

We denote diffusion processes also as

$$dX(t) = \sigma(t)dW(t) + \alpha(t)dt. \tag{5}$$

Note that

$$dX(t)dX(t) = \sigma^2(t)dW(t)dW(t) + \alpha^2(t)dtdt + \sigma(t)\alpha(t)dW(t)dt = \sigma^2(t)dt,$$

which means that the quadratic variation of the diffusion process (5) is given by

$$[X, X](t) = \int_0^t \sigma^2(s)\,ds, \quad t \geq 0.$$

Thus the stochastic process $\{\sigma^2(t)\}_{t\geq 0}$ measures the rate at which quadratic variations accumulates in time in the diffusion process $\{X(t)\}_{t\geq 0}$. Furthermore, assuming $\{\sigma(t)\}_{t\geq 0} \in \mathbb{L}^2[\mathcal{F}(t)]$, we have

$$\mathbb{E}[\int_0^t \sigma(s)dW(s)] = 0.$$

Hence the term $\int_0^t \alpha(s)ds$ is the only one contributing to the evolution of the average of $\{X(t)\}_{t\geq 0}$, which is the reason to call $\alpha(t)$ the drift of the diffusion process (if $\alpha = 0$ and $\{\sigma(t)\}_{t\geq 0} \in \mathbb{L}^2[\mathcal{F}(t)]$, the diffusion process is a martingale, as it follows by Theorem 1(ii)).

Finally, the integration along the paths of the diffusion process (5) is defined as

$$\int_0^t Y(s)dX(s) := \int_0^t Y(s)\sigma(s)dW(s) + \int_0^t Y(s)\alpha(s)ds, \tag{6}$$

for all $\{Y(t)\}_{t\geq 0} \in \mathcal{C}^0[\mathcal{F}(t)]$.

Ito's formula

Itô's integrals can often be computed, or at least reduced to standard integrals, by using the so-called **Itô formula**.

Theorem 2. *Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f = f(t,x)$, be a C^1 function such that $\partial_x^2 f$ is continuous and let $\{X(t)\}_{t\geq 0}$ be the diffusion process $dX(t) = \sigma(t)dW(t) + \alpha(t)dt$. Then* **Itô's formula** *holds:*

$$df(t,X(t)) = \partial_t f(t,X(t))\,dt + \partial_x f(t,X(t))\,dX(t) + \frac{1}{2}\partial_x^2 f(t,X(t))\,dX(t)dX(t), \tag{7}$$

i.e., using $dX(t)dX(t) = \sigma^2(t)\,dt$,

$$df(t,X(t)) = \partial_t f(t,X(t))\,dt + \partial_x f(t,X(t))\,(\sigma(t)dW(t) + \alpha(t)dt) + \frac{1}{2}\partial_x^2 f(t,X(t))\sigma^2(t)\,dt. \tag{8}$$

Recall that (8) is a shorthand for

$$f(t,X(t)) = f(0,X(0)) + \int_0^t (\partial_t f + \alpha(s)\partial_x f + \frac{1}{2}\sigma^2(s)\partial_x^2 f)(s,X(s))\,ds + \int_0^t \partial_x f(s,X(s))dW(s).$$

All integrals in the right hand side of the previous equation are well defined, as the integrand stochastic processes have continuous paths.

Examples.

1. Letting $X(t) = W(t)$ and $f(t,x) = x^2$, we obtain $d(W^2(t)) = 2W(t)dW(t) + dt$, hence
$$\int_0^t W(s)dW(s) = \frac{1}{2}W^2(t) - \frac{t}{2}$$
2. Letting $f(t,x) = tx$ we obtain $d(tW(t)) = W(t)dt + tdW(t)$, hence
$$\int_0^t sdW(s) = tW(t) - \int_0^t W(s)\,ds.$$
It can be shown that $\int_0^t W(s)\,ds \in \mathcal{N}(0, t^3/3)$.
3. Let $\{\theta(t)\}_{t\geq 0} \in \mathcal{C}^0[\mathcal{F}(t)]$ and define the stochastic process $\{Z(t)\}_{t\geq 0}$ by
$$Z(t) = \exp\left(-\int_0^t \theta(s)dW(s) - \frac{1}{2}\int_0^t \theta^2(s)ds\right). \tag{9}$$
Writing $Z(t) = e^{X(t)}$ and applying Itô's formula we obtain (see Exercise 4.6)
$$Z(t) = 1 - \int_0^t \theta(s)Z(s)\,dW(s). \tag{10}$$

Processes of the form (9) are fundamental in mathematical finance. In particular, it is important to know whether $\{Z(t)\}_{t\geq 0}$ is a martingale.

By (10) and Theorem 1(ii), $\{Z(t)\}_{t\geq 0}$ is a martingale if $\theta(t)Z(t) \in \mathbb{L}^2[\mathcal{F}(t)]$, which is however difficult in general to verify directly.

The following condition, known as **Novikov's condition**, is more useful in the applications, as it involves only the time-integral of the process $\{\theta(t)\}_{t\geq 0}$.

Theorem 3. *Let $\{\theta(t)\}_{t\geq 0} \in \mathcal{C}^0[\mathcal{F}(t)]$ satisfy*
$$\mathbb{E}[\exp(\frac{1}{2}\int_0^T \theta(t)^2\,dt)] < \infty, \quad \textit{for all } T > 0. \tag{11}$$
Then the stochastic process $\{Z(t)\}_{t\geq 0}$ given by
$$Z(t) = \exp\left(-\int_0^t \theta(s)dW(s) - \frac{1}{2}\int_0^t \theta^2(s)ds\right).$$
is a martingale relative the filtration $\{\mathcal{F}(t)\}_{t\geq 0}$.

In particular, the stochastic process $\{Z(t)\}_{t\geq 0}$ is a martingale when $\theta(t) = const$.

The following theorem contains the generalization of Itô's formula to functions of several random variables.

Theorem 4. *Let $f : \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ be a C^1 function such that $f = f(t,x)$ is twice continuously differentiable on the variable $x \in \mathbb{R}^N$. Let $\{X_1(t)\}_{t\geq 0}, \ldots, \{X_N(t)\}_{t\geq 0}$ be diffusion processes and let $X(t) = (X_1(t), \ldots, X_N(t))$. Then there holds:*

$$\begin{aligned} df(t,X(t)) = \partial_t f(t,X(t))\,dt &+ \sum_{i=1}^{N} \partial_{x^i} f(t,X(t))\,dX_i(t) \\ &+ \frac{1}{2}\sum_{i,j=1}^{N} \partial_{x^i}\partial_{x^j} f(t,X(t))\,dX_i(t)dX_j(t). \end{aligned} \tag{12}$$