Financial derivatives and PDE's
Lecture 5

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Exercise 2.8

Let $X \in \mathcal{N}(0,1)$ and $Y = X^2$. Show that $Y \in \chi^2(1)$.

Exercise 2.15

Derive the density of the geometric Brownian motion and use the result to show that $\mathbb{P}(S(t) = 0) = 0$, i.e., a stock whose price is described by a geometric Brownian motion cannot default.

Exercise 3.3

Prove the Schwarz inequality,

$$\mathbb{E}[XY] \leq \sqrt{\mathbb{E}[X^2]\mathbb{E}[Y^2]}, \tag{1}$$

for all random variables $X, Y \in L^2(\Omega)$.

Exercise 3.27

The purpose of this exercise is to show that the conditional expectation is the best estimator of a random variable when some information is given in the form of a sub-σ-algebra. Let $X \in L^1(\Omega)$ and $\mathcal{G} \subseteq \mathcal{F}$ be a sub-σ-algebra. Define $\mathrm{Err} = X - \mathbb{E}[X|\mathcal{G}]$. Show that $\mathbb{E}[\mathrm{Err}] = 0$ and

$$\mathrm{Var}[\mathrm{Err}] = \min_Y \mathrm{Var}[Y - X],$$

where the minimum is taken with respect to all $\mathcal{G}$-measurable random variables Y.

Exercise 4.4

Let $\{W_1(t)\}_{t\geq 0}$, $\{W_2(t)\}_{t\geq 0}$ be Brownian motions. Assume that there exists a constant $\rho \in [-1, 1]$ such that $dW_1(t)dW_2(t) = \rho dt$. Show that ρ is the correlation of the two Brownian motions at time t. Assuming that $\{W_1(t)\}_{t\geq 0}$, $\{W_2(t)\}_{t\geq 0}$ are independent, compute $\mathbb{P}(W_1(t) > W_2(s))$, for all $s, t > 0$.

Exercise 4.5

Consider the stochastic process $\{X(t)\}_{t\geq 0}$ defined by $X(t) = W(t)^3 - 3tW(t)$. Show that $\{X(t)\}_{t\geq 0}$ is a martingale and find a process $\{\Gamma(t)\}_{t\geq 0}$ adapted to $\{\mathcal{F}_W(t)\}_{t\geq 0}$ such that

$$X(t) = X(0) + \int_0^t \Gamma(s)dW(s).$$

(The existence of the process $\{\Gamma(t)\}_{t\geq 0}$ is ensured by Theorem 4.5(v).)

Exercise 4.6

Let $\{\theta(t)\}_{t\geq 0} \in \mathcal{C}^0[\mathcal{F}(t)]$ and define the stochastic process $\{Z(t)\}_{t\geq 0}$ by

$$Z(t) = \exp\left(-\int_0^t \theta(s)dW(s) - \frac{1}{2}\int_0^t \theta^2(s)ds\right).$$

Show that

$$Z(t) = 1 - \int_0^t \theta(s)Z(s)\,dW(s).$$