Financial derivatives and PDE's
Lecture 6

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Remarks

- More information on the financial concepts introduced in this lecture can be found in the text *Basic financial concepts*, available on the course homepage.
- Unless otherwise stated, it is assumed throughout the course that *assets pay no dividend.*

Throughout the course we assume that the following is given:

- a probability space $(\Omega, \mathcal{F}, \mathbb{P})$,
- a $\mathbb{P}$-Brownian motion $\{W(t)\}_{t\geq 0}$ and
- a non-anticipating filtration $\{\mathcal{F}(t)\}_{t\geq 0}$.

Also, we are only interested on what happens in some finite time interval $t \in [0, T]$ and in this case it is usually assumed (even if not always necessary) that

$$\mathcal{F}(T) = \mathcal{F} = \mathcal{F}_W(T)$$

i.e., the full information is revealed at time T and it is contained in the Brownian motion.

All variables in financial mathematics are represented by stochastic processes. The most obvious example is the **price** of financial assets.

All stochastic processes describing financial variables in this course are assumed to be in $\mathcal{C}^0[\mathcal{F}_W(t)]$.

Stock price

The price per share a time t of a stock will be denoted by $S(t)$.

Most commonly $S(t) > 0$, for all $t \geq 0$, however in some models it is possible that $S(t) = 0$ with positive probability (risk of default).

$\{S(t)\}_{t\geq 0}$ is a stochastic process. If we have several stocks, we shall denote their price by $\{S_1(t)\}_{t\geq 0}$, $\{S_2(t)\}_{t\geq 0}$, etc.

In this course we assume that $\{S(t)\}_{t\geq 0}$ is given by a generalized geometric Brownian motion, defined as follows.

Generalized geometric Brownian motion

Given two stochastic processes $\{\alpha(t)\}_{t\geq 0}, \{\sigma(t)\}_{t\geq 0} \in \mathcal{C}^0[\mathcal{F}_W(t)]$, the stochastic process $\{S(t)\}_{t\geq 0} \in \mathcal{C}^0[\mathcal{F}_W(t)]$ given by

$$S(t) = S(0) \exp\left(\int_0^t \alpha(s)ds + \int_0^t \sigma(s)dW(s)\right) \tag{1}$$

is called **generalized geometric Brownian motion** with **mean of log-return** (or **log-drift**) $\{\alpha(t)\}_{t\geq 0}$ and **volatility** $\{\sigma(t)\}_{t\geq 0}$.

Since

$$d(\log S(t)) = \alpha(t)\, dt + \sigma(t)\, dW(t),$$

then the **log-price** of the stock is a diffusion process with drift rate $\alpha(t)$ and diffusion rate $\sigma(t)$ (i.e., $\sigma(t)^2$ is the rate of quadratic variation of $\log S(t)$).

When $\alpha(t) = \alpha \in \mathbb{R}$ and $\sigma(t) = \sigma > 0$ are deterministic constant, the process (1) is called geometric Brownian motion.

Since

$$S(t) = S(0)e^{X(t)}, \quad dX(t) = \alpha(t)dt + \sigma(t)dW(t),$$

then Itô's formula gives

$$
\begin{aligned}
dS(t) &= S(0)e^{X(t)}dX(t) + \frac{1}{2}S(0)e^{X(t)}dX(t)dX(t) \\
&= S(t)\alpha(t)dt + S(t)\sigma(t)dW(s) + \frac{1}{2}\sigma^2(t)S(t)dt
\end{aligned}
$$

that is

$$
dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW(t), \tag{2}
$$

where

$$
\mu(t) = \alpha(t) + \frac{1}{2}\sigma^2(t)
$$

Prescribing $S(t)$ as in (1) with given $\{\alpha(t)\}_{t\geq 0}, \{\sigma(t)\}_{t\geq 0} \in \mathcal{C}^0[\mathcal{F}_W(t)]$ is equivalent to prescribing $S(t)$ as in (2) with given $\{\mu(t)\}_{t\geq 0}, \{\sigma(t)\}_{t\geq 0} \in \mathcal{C}^0[\mathcal{F}_W(t)]$. The latter will be used most commonly in this course.

In the presence of several stocks, it is reasonable to assume that each of them introduced a new source of randomness in the market.

Thus, when dealing with N stocks, we assume the existence of N independent Brownian motions $\{W_1(t)\}_{t\geq 0}, \dots, \{W_N(t)\}_{t\geq 0}$ and model the evolution of the stocks prices

$$
\{S_1(t)\}_{t\geq 0}, \dots, \{S_N(t)\}_{t\geq 0}
$$

by the following N-**dimensional generalized geometric Brownian motion**:

$$
dS_k(t) = \left(\mu_k(t) + \sum_{j=1}^{N} \sigma_{kj}(t)dW_j(t)\right) S_k(t) \tag{3}
$$

for some stochastic processes $\{\mu_k(t)\}_{t\geq 0}, \{\sigma_{kj}(t)\}_{t\geq 0} \in \mathcal{C}^0[\mathcal{F}_W(t)]$, $j, k = 1, \dots, N$.

Financial derivative

A **financial derivative** (or derivative security) is a contract whose value depends on the performance of one (or more) other asset(s), which is called the **underlying asset**.

There exist various types of financial derivatives, the most common being options, futures, forwards and swaps.

Financial derivatives can be traded **over the counter** (OTC), or in a regularized **exchange market**.

In the former case, the contract is stipulated between two individual investors, who agree upon the conditions and the price of the contract. In particular, the same derivative (on the same asset, with the same parameters) can have two different prices over the counter.

Derivatives traded in the market, on the contrary, are standardized contracts. Anyone, after a proper authorization, can make offers to buy or sell derivatives in the market, in a way much similar to how stocks are traded. Let us see some examples of financial derivatives (we shall introduce more examples later).

Call and put options

A **call option** is a contract between two parties, the buyer (or **owner**) of the call and the seller (or **writer**) of the call.

The contract gives to the buyer the right, but not the obligation, to buy the underlying asset at some future time for a price agreed upon today, which is called **strike price** of the call.

If the buyer can exercise this option only at some given time $t = T > 0$ (where $t = 0$ corresponds to the time at which the contract is stipulated) then the call option is called **European**, while if the option can be exercised at any time in the interval $(0, T]$, then the option is called **American**.

The time $T > 0$ is called **maturity time**, or **expiration date** of the call.

The seller of the call is obliged to sell the asset to the buyer (at the strike price) if the latter decides to exercise the option.

If the option to buy in the definition of a call is replaced by the option to sell, then the option is called a **put option**.

In exchange for the option, the buyer must pay a **premium** to the seller.

Suppose that the option is a European option with strike price K, maturity time T and premium Π_0 on a stock with price $S(t)$ at time t. In which case is it then convenient for the buyer to exercise the call? Let us define the **payoff** of a European call as

$$Y = (S(T) - K)_+ := \max(0, S(T) - K) \quad \text{(call)};$$

similarly for a European put we set

$$Y = (K - S(T))_+ \quad \text{(put)}.$$

Note that Y is a random variable, because it depends on the random variable $S(T)$.

Clearly, if $Y > 0$ it is more convenient for the buyer to exercise the option rather than buying/selling the asset on the market.

A call (resp. put) is said to be **in the money** at time t if $S(t) > K$ (resp. $S(t) < K$).

The call (resp. put) is said to be **out of the money** if $S(t) < K$ (resp. $S(t) > K$).

If $S(t) = K$, the (call or put) option is said to be **at the money** at time t.

European derivatives

European call and put options are examples of more general contracts called **European derivatives**.

Given a function $g : (0, \infty) \to \mathbb{R}$, a **standard European derivative** with pay-off $Y = g(S(T))$ and maturity time $T > 0$ is a contract that pays to its owner the amount Y at time $T > 0$.

Here $S(T)$ is the price of the underlying asset (which we take to be a stock) at time T.

The function g is called **pay-off function** of the derivative, while $Y(t) = g(S(t))$ is called **intrinsic value** of the derivative.

The term "European" refers to the fact that the contract cannot be exercised before time T, while the term "standard" refers to the fact that the pay-off depends only on the price of the underlying at time T.

The pay-off of non-standard (or **exotic**) European derivatives depends on the path of the asset price during the interval $[0, T]$. For example, the pay-off of an **Asian call** is given by

$$Y = \left(\frac{1}{T} \int_0^T S(t)\, dt - K\right)_+.$$

The price at time t of a European derivative (standard or not) with pay-off Y and expiration date T will be denoted by $\Pi_Y(t)$.

Hence $\{\Pi_Y(t)\}_{t\in[0,T]}$ is a stochastic process.

A **standard American derivative** with pay-off function g is a contract which can be exercised at any time $t \in (0,T]$ prior or equal to its maturity and that, upon exercise, pays the amount $g(S(t))$ (i.e., the intrinsic value) to the holder of the derivative.

Non-standard American derivatives are defined similarly as the European ones but with the further option of earlier exercise. In this course we are mostly concerned with European derivatives, but in the laset we also discuss briefly some properties of American call/put options.

Zero-coupon bonds

A **zero-coupon bond** (ZCB) with maturity T and **face value** 1 is the European derivative that pays the constant pay-off $Y = 1$ at time T.

Let $B(t,T)$, or simply $B(t)$, denote the value at time t of the ZCB.

The **continuously compounded**, or **short**, **interest rate** of the ZCB is defined as

$$r(t) = \frac{d}{dt}\log B(t),$$

hence, integrating in the interval $[t,T]$, we can write the value of the ZCB as

$$B(t) = e^{-\int_t^T r(s)\,ds}.$$

Thus, given a model for the interest rate, i.e., given a stochastic process $\{r(t)\}_{t\geq 0} \in \mathcal{C}^0[\mathcal{F}_W(t)]$, the value of the ZCB satisfies

$$dB(t) = -B(t)r(t)dt,$$

If the institution issuing the ZCB bears no risk of default, the ZCB is called a **risk-free asset**.

In this course it is assumed that all risk-free assets have the same interest rate $r(t)$, which is called **risk-free rate**.

The discount process

Let $\{r(t)\}_{t\geq 0}$ be a stochastic process modeling the risk-free rate. Denote by $B(t)$ the value at time t of a risk-free asset (e.g., a risk-free ZCB). The stochastic process $\{D(t)\}_{t\geq 0}$ given by

$$D(t) = \frac{B(0)}{B(t)} = \exp\left(-\int_0^t r(s)\,ds\right) \tag{4}$$

is called the **discount process**.

If $\tau < t$ and $X(t)$ denotes the price of an asset at time t, the quantity $D(t)X(t)/D(\tau)$, is called the t-price of the asset discounted at time τ.

When $\tau = 0$ we refer to $D(t)X(t)/D(0) = D(t)X(t) = X^*(t)$ simply as the **discounted price** of the asset.

For instance, the discounted (at time $t = 0$) price of a stock with price $S(t)$ at time t is given by $S^*(t) = D(t)S(t)$ and has the following meaning:

$S^*(t)$ is the amount that should be invested on the money market at time $t = 0$ in order that the value of this investment at time t replicates the value of the stock at time t. Notice that $S^*(t) < S(t)$ when $r(t) > 0$.

The discounted price of the stock measures, roughly speaking, the loss in the stock value due to the "time-devaluation" of money expressed by the ratio $B(0)/B(t)$.

Portfolio

The portfolio of an investor is the set of all assets in which the investor is trading. Mathematically it is described by a collection of N stochastic processes

$$\{h_1(t)\}_{t\geq 0}, \{h_2(t)\}_{t\geq 0}, \dots, \{h_N(t)\}_{t\geq 0} \in \mathcal{C}^0[\mathcal{F}_W(t)],$$

where $h_k(t)$ represents the number of shares of the asset k at time t in the investor portfolio.

If $h_k(t)$ is positive, resp. negative, the investor has a long, resp. short, position on the asset k at time t.

If $\Pi_k(t)$ denotes the value of the asset k at time t, then $\{\Pi_k(t)\}_{t\geq 0}$ is a stochastic process; the **portfolio value** is the stochastic process $\{V(t)\}_{t\geq 0}$ given by

$$V(t) = \sum_{k=1}^{N} h_k(t)\Pi_k(t).$$

For modeling purposes, it is convenient to assume that an investor can trade any fraction of shares of the assets, i.e., $h_k(t) : \Omega \to \mathbb{R}$, rather than $h_k(t) : \Omega \to \mathbb{Z}$

Markets

A market in which the objects of trading are N risky assets (e.g., stocks) and M risk-free assets is said to be "$N + M$ dimensional".

In Most of this course we shall focus on the case of **1+1 dimensional markets** in which the risky asset is, typically, a stock.

We assume that the price of the stock follows the generalized geometric Brownian motion

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW(t), \tag{5}$$

while the value of the risk-free asset is given by

$$dB(t) = B(t)r(t)dt, \tag{6}$$

where $\{r(t)\}_{t\geq 0}$ is the risk-free rate of the money market.

Moreover we assume that the **market parameters** $\{\mu(t)\}_{t\geq 0}$, $\{\sigma(t)\}_{t\geq 0}$, $\{r(t)\}_{t\geq 0}$ have continuous paths a.s. and are adapted to the filtration $\{\mathcal{F}_W(t)\}_{t\geq 0}$, i.e., they belong to $\mathcal{C}^0[\mathcal{F}_W(t)]$.

A portfolio process invested in this market is a stochastic process $\{h_S(t), h_B(t)\}_{t\geq 0} \in \mathcal{C}^0[\mathcal{F}_W(t)]$, where $h_S(t)$ is the number of shares of the stock and $h_B(t)$ the number of shares of the risk-free asset in the portfolio at time t.

The value of the portfolio is given by

$$V(t) = h_S(t)S(t) + h_B(t)B(t). \tag{7}$$

Self-financing portfolio

Consider a portfolio $\{h_S(t), h_B(t)\}_{t\geq 0} \in \mathcal{C}^0[\mathcal{F}_W(t)]$ invested in a 1+1-dimensional market.

We say that the portfolio is self-financing if purchasing more shares of one asset is possible only by selling shares of the other asset for an equivalent value (and not by infusing new cash into the portfolio), and, conversely, if any cash obtained by selling one asset is immediately re-invested to buy shares of the other asset (and not withdrawn from the portfolio).

To translate this condition into a mathematical formula, assume that (h_S, h_B) is the investor position on the stock and the risk-free asset during the "infinitesimal" time interval $[t, t+\delta t)$.

Let $V^-(t+\delta t)$ be the value of this portfolio immediately before the time $t+\delta t$ at which the position is changed, i.e.,

$$V^-(t+\delta t) = \lim_{u\to t+\delta t} h_S S(u) + h_B B(u) = h_S S(t+\delta t) + h_B B(t+\delta t),$$

where we used the continuity in time of the assets price.

At the time $t+\delta t$, the investor sells/buys shares of the assets. Let (h'_S, h'_B) be the new position on the stock and the risk-free asset.

Then the value of the portfolio at time $t+\delta t$ is given by

$$V(t+\delta t) = h'_S S(t+\delta t) + h'_B B(t+\delta t).$$

The difference $V(t+\delta t) - V^-(t+\delta t)$, if not zero, corresponds to cash withdrawn or added to the portfolio as a result of the change in the position on the assets. In a self-financing portfolio, however, this difference must be zero. We obtain

$$V(t+\delta t) - V^-(t+\delta t) = 0 \Leftrightarrow (h_S - h'_S)S(t+\delta t) + (h_B - h'_B)B(t+\delta t) = 0.$$

Hence, the change of the portfolio value in the interval $[t, t+\delta t]$ is given by

$$\delta V = V(t+\delta t) - V(t) = h'_S S(t+\delta t) + h'_B B(t+\delta t) - (h_S S(t) + h_B B(t)) = h_S \delta S + h_B \delta B,$$

where $\delta S = S(t+\delta t) - S(t)$, and $\delta B = B(t+\delta t) - B(t)$ are the changes of the assets value in the interval $[t, t+\delta t]$. This discussion leads to the following definition.

Definition 1. *A portfolio process* $\{h_S(t), h_B(t)\}_{t\geq 0} \in \mathcal{C}^0[\mathcal{F}_W(t)]$ *invested in the* $1+1$*-dimensional market* (5)-(6) *is said to be* **self-financing** *if its value process* $\{V(t)\}_{t\geq 0}$ *satisfies*

$$dV(t) = h_S(t)dS(t) + h_B(t)dB(t). \tag{8}$$

The owner of a self-financing portfolio makes a profit in the time interval $[0, T]$ if $V(T) > V(0)$, while if $V(T) < V(0)$ the investor incurs in a loss.

Exercise 4.10

Show that given a diffusion process $\{h_S(t)\}_{t\geq 0}$, it is always possible to find a diffusion process $\{h_B(t)\}_{t\geq 0}$ such that the portfolio process $\{h_S(t), h_B(t)\}_{t\geq 0}$ is self-financing.

Arbitrage portfolio

We now introduce the important definition of arbitrage portfolio.

Definition 2. *A self-financing portfolio process in the interval* $[0, T]$ *is said to be an* **arbitrage portfolio** *if its value* $\{V(t)\}_{t\in[0,T]}$ *satisfies the following properties:*

(i) $V(0) = 0$ *almost surely;*

(ii) $V(T) \geq 0$ *almost surely;*

(iii) $\mathbb{P}(V(T) > 0) > 0$.

Hence a self-financing arbitrage portfolio is a risk-free investment in the interval $[0, T]$ which requires no initial wealth and with a positive probability to give profit.

We remark that the arbitrage property depends on the probability measure $\mathbb{P}$. However, it is clear that if two measures $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent, then the arbitrage property is satisfied with respect to $\mathbb{P}$ if and only if it is satisfied with respect to $\widetilde{\mathbb{P}}$.

The guiding principle to devise theoretical models for asset prices in financial mathematics is to ensure that one cannot set-up an arbitrage portfolio by investing on these assets, in which case the market is said to be **arbitrage-free**.

Hedging portfolio

Suppose that at time t a European derivative with pay-off Y at the time of maturity $T > t$ is sold for the price $\Pi_Y(t)$.

An important problem in financial mathematics is to find a strategy for how the seller should invest the premium $\Pi_Y(t)$ of the derivative in order to **hedge** the derivative, i.e., in order to ensure that the portfolio value of the seller at time T is enough to pay-off the buyer of the derivative.

We assume that the seller invests the premium of the derivative only on the 1+1 dimensional market consisting of the underlying stock and the risk-free asset (Δ-hedging).

Definition 3. *Consider the European derivative with pay-off Y and time of maturity T, where we assume that Y is $\mathcal{F}_W(T)$-measurable. A portfolio process $\{h_S(t), h_B(t)\}_{t\geq 0} \in \mathcal{C}^0[\mathcal{F}_W(t)]$ invested in the underlying stock and the risk-free asset is said to be an* **hedging portfolio** *if its value of the portfolio satisfies $V(T) = Y$.*

The main questions that we want to answer are:

1) What is a reasonable "fair" price for the European derivative at time $t \in [0, T]$?

2) What investment strategy (on the underlying stock and the risk-free asset) should the seller undertake in order to hedge the derivative?