Financial derivatives and PDE's
Lecture 8

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Notation

Given $T > 0$, we denote by $\mathcal{D}_T$ the open region in the (t, x)-plane given by

$$\mathcal{D}_T = \{t \in (0, T), x \in \mathbb{R}\} = (0, T) \times \mathbb{R}.$$

The closure and the boundary of $\mathcal{D}_T$ are given respectively by

$$\overline{\mathcal{D}_T} = [0, T] \times \mathbb{R}, \quad \partial\mathcal{D}_T = \{t = 0, x \in \mathbb{R}\} \cup \{t = T, x \in \mathbb{R}\}.$$

Similarly we denote $\mathcal{D}_T^+$ the open region

$$\mathcal{D}_T^+ = \{t \in (0, T), x > 0\} = (0, T) \times (0, \infty),$$

whose closure and boundary are given by

$$\overline{\mathcal{D}_T^+} = [0, T] \times [0, \infty), \quad \partial\mathcal{D}_T^+ = \{t = 0, x \geq 0\} \cup \{t = T, x \geq 0\} \cup \{t \in [0, T], x = 0\}.$$

Moreover we shall employ the following notation for functions spaces: For $\mathcal{D} = \mathcal{D}_T$ or $\mathcal{D}_T^+$,

- $C^k(\mathcal{D})$ is the space of k-times continuously differentiable functions $u : \mathcal{D} \to \mathbb{R}$;
- $C^{1,2}(\mathcal{D})$ is the space of functions $u \in C^1(\mathcal{D})$ such that $\partial_x^2 u \in C(\mathcal{D})$;
- $C^k(\overline{\mathcal{D}})$ is the space of functions $u : \overline{\mathcal{D}} \to \mathbb{R}$ such that $u \in C^k(\mathcal{D})$ and the partial derivatives of u up to order k extend continuously on $\overline{\mathcal{D}}$.
- $C_c^k(\mathbb{R}^n)$ is the space of k-times continuously differentiable functions $u : \mathbb{R}^n \to \mathbb{R}$ with compact support. We also let $C_c^\infty(\mathbb{R}^n) = \cap_{k\in\mathbb{N}} C_c^k(\mathbb{R}^n)$

A function $u : \mathcal{D} \to \mathbb{R}$ is **uniformly bounded** if there exists $C_T > 0$ such that $|u(t, x)| \leq C_T$, for all $(t, x) \in \mathcal{D}$. Unless otherwise stated, all functions are real-valued.

1 Kolmogorov PDE's

Consider the SDE

$$dX(t) = \alpha(t, X(t))\, dt + \beta(t, X(t))\, dW(t) \tag{1}$$

with initial value $X(s, \omega) = x$ at time $t = s$. Recall that we denote $\{X(t; s, x\}_{t\geq s}$ the solution, which exists and is unique under the assumptions

$$|\alpha(t, x)| + |\beta(t, x)| \leq C_T(1 + |x|), \tag{2a}$$

$$|\alpha(t, x) - \alpha(t, y)| + |\beta(t, x) - \beta(t, y)| \leq D_T|x - y|, \tag{2b}$$

Most financial variables are represented by stochastic processes solving (systems of) SDE's.

In this context, a problem which recurs often is to find a function f such that the process $\{Y(t)\}_{t\geq 0}$ given by $Y(t) = f(t, X(t))$ is a martingale, where $\{X(t)\}_{t\geq 0}$ is the global solution of (1) with initial value $X(0) = x$. To this regard we have the following result.

Theorem 1. *Let $T > 0$ and $u : \overline{\mathcal{D}_T} \to \mathbb{R}$ such that $u \in C^{1,2}(\mathcal{D}_T)$ and $\partial_x u$ is uniformly bounded. Assume that u satisfies the partial differential equation*

$$\partial_t u + \alpha(t, x)\partial_x u + \frac{1}{2}\beta(t, x)^2\partial_x^2 u = 0 \tag{3}$$

in the region $\mathcal{D}_T$. Assume also that α, β satisfy the conditions (2) and let $\{X(t)\}_{t\geq 0}$ be the unique global solution of (1) with initial value $X(0) = x$. The stochastic process $\{u(t, X(t))\}_{t\in[0,T]}$ is a martingale and satisfies

$$u(t, X(t)) = u(0, x) + \int_0^t \beta(\tau, X(\tau))\partial_x u(\tau, X(\tau))\, dW(\tau), \quad t \in [0, T]. \tag{4}$$

Proof. By Itô's formula we find

$$du(t, X(t)) = (\partial_t u + \alpha\partial_x u + \frac{\beta^2}{2}\partial_x^2 u)(t, X(t))\, dt + (\beta\partial_x u)(t, X(t))\, dW(t).$$

As u solves (3), then $du(t, X(t)) = (\beta\partial_x u)(t, X(t))\, dW(t)$, which is equivalent to (4) (because $u(0, X(0)) = u(0, x)$). As $\partial_x u$ is uniformly bounded, there exists a constant $C_T > 0$ such that $|\partial_x u(t, x)| \leq C_T$ and so, due also to (2a), the process $Y(t) = \beta\partial_x u(t, X(t))$ satisfies $|Y(t)| \leq C_T(1 + |X(t)|)$. Since $\{X(t)\}_{t\geq 0} \in \mathbb{L}^2(\mathcal{F}(t))$, then $\{Y(t)\}_{t\geq 0} \in \mathbb{L}^2(\mathcal{F}(t))$ as well and so the Itô integral in the right hand side of (4) is a martingale. This concludes the proof of the theorem. $\square$

Definition 1. *The partial differential equation (PDE)* (3) *is called the* **backward Kolmogorov equation** *associated to the SDE* (1)*. We say that* $u : \overline{\mathcal{D}_T} \to \mathbb{R}$ *is a* **strong solution** *of* (3) *in the region* $\mathcal{D}_T$ *if* $u \in C^{1,2}(\mathcal{D}_T)$*,* $\partial_x u$ *is uniformly bounded and* u *solves* (3) *for all* $(t,x) \in \mathcal{D}_T$*. Similarly, replacing* $\mathcal{D}_T$ *with* $\mathcal{D}_T^+$*, one defines strong solutions of* (3) *in the region* $\mathcal{D}_T^+$

Exercise 5.6 Derive the backward Kolmogorov PDE associated to the system of SDE's

$$dX_i(t) = \alpha_i(t, X_1(t), X_2(t))\, dt + \sum_{j=1,2} \beta_{ij}(t, X_1(t), X_2(t)) dW_j(t), \tag{5a}$$

$$X_i(s) = x_i, \quad i = 1, 2 \tag{5b}$$

when the Brownian motions $\{W_1(t)\}_{t\geq 0}$, $\{W_2(t)\}_{t\geq 0}$ have constant correlation $\rho \in [-1, 1]$. HINT: Remember that $dW_1(t)dW_2(t) = \rho\, dt$.

The statement of Theorem 1 rises the question of whether the backward Kolmogorov PDE admits in general strong solutions.

This problem is discussed, with different degrees of generality, in any textbook on PDE's.

Here we are particularly interested in which conditions ensure the *uniqueness* of the strong solution. To this regard we have the following theorem.

Theorem 2. *Assume that $\alpha, \beta \in C^0([0,\infty)\times\mathbb{R})$ satisfy* (2a)-(2b) *for all $T > 0$ and $(t,x) \in [0,T]\times\mathbb{R}$ and let $\{X(t;x,s)\}_{t\geq s}$ be the unique global solution of* (1) *with initial value $X(s) = x$. Let $g \in C^2(\mathbb{R})$, resp. $g \in C^2([0,\infty))$ such that g' is uniformly bounded. The backward Kolmogorov PDE*

$$\partial_t u + \alpha(t,x)\partial_x u + \frac{1}{2}\beta(t,x)^2\partial_x^2 u = 0, \quad (t,x) \in \mathcal{D}_T, \quad \textit{resp. } (t,x) \in \mathcal{D}_T^+, \tag{6}$$

with the **terminal condition**

$$\lim_{t\to T} u(t,x) = g(x), \quad \textit{for all } x \in \mathbb{R}, \textit{ resp. } x > 0, \tag{7}$$

admits at most one strong solution. Moreover, when it exists, the strong solution is given by the **Feynman-Kac** *formula:*

$$u_T(t,x) = \mathbb{E}[g(X(T;t,x))], \quad 0 \leq t \leq T. \tag{8}$$

Proof. Let v be a strong solution and set $Y(\tau) = v(\tau, X(\tau;t,x))$, for $t \leq \tau \leq T$. By Itô's formula and using that v solves (6) we find $dY(\tau) = \beta\partial_x v(\tau, X(\tau;t,x))dW(\tau)$. Hence

$$v(T, X(T;t,x)) - v(t, X(t;t,x)) = \int_t^T \beta\partial_x v(\tau, X(\tau;t,x))dW(\tau). \tag{9}$$

Moreover $v(T, X(T;t,x)) = g(X(T;t,x))$, $v(t, X(t;t,x)) = v(t,x)$ and in addition, by (2a) and the fact that of $\partial_x v$ is uniformly bounded, the Itô integral in the right hand side of (9) is a martingale. Hence taking the expectation we find $v(t,x) = \mathbb{E}[g(T, X(T;t,x))] = u(t,x)$. $\square$

Remarks

- The function (8) is indeed the strong solution of the Kolmogorov PDE in the whole space $x \in \mathbb{R}$ under quite general conditions on the terminal value g and the coefficients α, β. The case when the problem is posed on the half-space $x > 0$ is however more subtle, as we shall see later for the Kolmogorov PDE associated to the CIR process.
- The conditions on the function g in Theorem 2 can be considerably weakened. In particular the theorem still holds if one chooses g to be the pay-off function of call (or put) options, i.e., $g(x) = (x - K)_+$, although of course in this case the solution does not have a smooth extension on the terminal time boundary $t = T$.

- It is often convenient to study the backward Kolmogorov PDE with an initial, rather than terminal, condition. To this purpose it suffices to make the change of variable $t \to T - t$ in (6). Letting $\bar{u}(t,x) = u(T-t,x)$, we now see that $\bar{u}$ satisfies the PDE

$$-\partial_t \bar{u} + \alpha(T-t,x)\partial_x \bar{u} + \frac{1}{2}\beta(T-t,x)^2 \partial_x^2 \bar{u} = 0, \tag{10}$$

with initial condition $\bar{u}(0,x) = g(x)$

- It is possible to define other concepts of solution to the backward Kolmogorov PDE than the strong one, e.g., weak solution, entropy solution, etc. In general these solutions are not uniquely characterized by their terminal value. In these notes we only consider strong solutions, which, as proved in Theorem 2, are uniquely determined by (7).

Exercise 5.7

Derive the backward Kolmogorov PDE in the form (10) associated to the linear SDE

$$dX(t) = (a + bX(t))\,dt + \gamma dW(t), \quad t \geq s, \quad X(s) = x \tag{11}$$

and verify that the transition density of this process, namely

$$p_*(t,x,y) = e^{\frac{(y-m(t,x))^2}{2\Delta(t)^2}} \frac{1}{\sqrt{2\pi\Delta(t)^2}}, \; m(t,x) = xe^{bt} + \frac{a}{b}(e^{bt}-1), \quad \Delta(t)^2 = \frac{\gamma^2}{2b}(e^{2bt}-1). \tag{12}$$

is a solution for all $y \in \mathbb{R}$. Find also the strong solution of the Kolmogorov PDE (in the form (10)) in the region $\mathcal{D}_T^+$ and with initial condition $u(0,x) = e^{-x}$. HINT: For the second part use the *ansatz* $u(t,x) = e^{-xA(t)+B(t)}$.

As suggested by the previous exercise, the study of the backward Kolmogorov equation is also important to derive the transition density for stochastic processes solutions of SDE's.

In fact, it can be shown that when $\{X(t)\}_{t\geq s}$ admits a smooth transition density, then the latter coincides with the **fundamental solution** of the backward Kolmogorov equation.

To state the result, let us denote by $\delta(x-y)$ the δ**-distribution** centered in $y \in \mathbb{R}$, i.e., the distribution satisfying

$$\int_{\mathbb{R}} \psi(x)\delta(x-y)\,dx = \psi(y), \quad \text{for all } \psi \in C_c^\infty(\mathbb{R}).$$

A sequence of measurable functions $(g_n)_{n\in\mathbb{N}}$ is said to converge to $\delta(x-y)$ in the sense of distributions if

$$\lim_{n\to\infty} \int_{\mathbb{R}} g_n(x)\psi(x)\,dx \to \psi(y), \quad \text{as } n \to \infty, \text{ for all } \psi \in C_c^\infty(\mathbb{R}).$$

Theorem 3. . *Let $\{X(t; s, x)\}_{t\geq s}$ be the global solution of* (1) *with initial value $X(s) = x$; recall that this solution is a Markov stochastic process.*

(i) If $\{X(t; s, x)\}_{t\geq s}$ admits a transition density $p(t, s, x, y)$ which is C^1 in the variable s and C^2 in the variable x, then $p(t, s, x, y)$ solves the backward Kolmogorov PDE

$$\partial_s p + \alpha(s, x)\partial_x p + \frac{1}{2}\beta(s, x)^2\partial_x^2 p = 0, \quad 0 < s < t, \ x \in \mathbb{R}, \tag{13}$$

with terminal value

$$\lim_{s\to t} p(t, s, x, y) = \delta(x - y). \tag{14}$$

(ii) If $\{X(t; s, x)\}_{t\geq s}$ admits a transition density $p(t, s, x, y)$ which is C^1 in the variable t and C^2 in the variable y then $p(t, s, x, y)$ solves the **forward Kolmogorov** *(or* **Fokker-Planck***) PDE:*

$$\partial_t p + \partial_y(\alpha(t, y)p) - \frac{1}{2}\partial_y^2(\beta(t, y)^2 p) = 0, \quad t > s, \ x \in \mathbb{R}, \tag{15}$$

with initial value

$$\lim_{t\to s} p(t, s, x, y) = \delta(x - y). \tag{16}$$

Example.

Recall that when the functions α, β in (1) are time-independent, then the Markovian stochastic process $\{X(t; s, x)\}_{t\geq s}$ is homogeneous and therefore the transition density, when it exists, has the form $p(t, s, x, y) = p_*(t - s, x, y)$.

By the change of variable $s \to t - s = \tau$ in (13), and by (15), we find that $p_*(\tau, x, y)$ satisfies

$$-\partial_\tau p_* + \alpha(x)\partial_x p_* + \frac{1}{2}\sigma(x)^2\partial_x^2 p_* = 0, \tag{17}$$

as well as

$$\partial_\tau p_* + \partial_y(\alpha(y)p_*) - \frac{1}{2}\partial_y^2(\sigma(y)^2 p_*) = 0, \tag{18}$$

with the initial condition $p_*(0, x, y) = \delta(x - y)$.

For example the Brownian motion is a Markov process with transition density

$$p_*(\tau, x, y) = \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{(y-x)^2}{2\tau}}. \tag{19}$$

In this case, (17) and (18) both reduce to the heat equation $-\partial_\tau p_* + \frac{1}{2}\partial_x^2 p_* = 0$.

It is straightforward to verify that (19) satisfies the heat equation for $(\tau, x) \in (0, \infty) \times \mathbb{R}$.

Now we show that, as claimed in Theorem 3, the initial condition $p_*(0, x, y) = \delta(x - y)$ is also verified, that is

$$\lim_{\tau \to 0} \int_{\mathbb{R}} p_*(\tau, x, y)\psi(y)\, dy = \psi(x), \quad \text{for all } \psi \in C_c^\infty(\mathbb{R}) \text{ and } x \in \mathbb{R}.$$

Indeed with the change of variable $y = x + \sqrt{\tau} z$, we have

$$\int_{\mathbb{R}} p_*(\tau, x, y)\psi(y)\, dy = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{z^2}{2}} \psi(x + \sqrt{\tau} z)\, dz \to \psi(0) \int_{\mathbb{R}} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}} = \psi(0),$$

as claimed.