Financial derivatives and PDE's
Lecture 2

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January 20^{th}, 2021

1 Expectation

The **expectation** of a random variable X is denoted by $\mathbb{E}[X]$. It represents an estimate on the average value of X.

For simple random variables it is defined as

$$\mathbb{E}[\sum_{k=1}^{N} a_k \mathbb{I}_{A_k}] = \sum_k a_k \mathbb{P}(A_k).$$

If $X :\to \mathbb{R}$ is a random variable with density f_X, the expectation is given by

$$\mathbb{E}[X] = \int_{\mathbb{R}} f_X(x)\, dx. \tag{1}$$

For instance

$$X \in \mathcal{N}(m, \sigma^2) \Rightarrow \mathbb{E}[X] = m, \quad X \in \chi^2(\delta, \beta) \Rightarrow \mathbb{E}[X] = \delta + \beta.$$

The set of all random variables with finite expectation is denoted by $L^1(\Omega)$.

The expectation satisfies the following properties:

Theorem 1 (Properties of the expectation). *Let $X, Y \in L^1(\Omega)$.*

(i) Linearity: For all $\alpha, \beta \in \mathbb{R}$, $\mathbb{E}[\alpha X + \beta Y] = \alpha \mathbb{E}[X] + \beta \mathbb{E}[Y]$;

(ii) If $X \leq Y$ a.s. then $\mathbb{E}[X] \leq \mathbb{E}[Y]$;

(iii) If $X \geq 0$ a.s., then $\mathbb{E}[X] = 0$ if and only if $X = 0$ a.s.;

(iv) If X, Y are independent, then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$;

(v) If X has the density f_X, then

$$\mathbb{E}[g(X)] = \int_{\mathbb{R}} g(x) f_X(x)\, dx$$

for all measurable functions g such that $g(X) \in L^1(\Omega)$;

(vi) If X, Y have the joint density $f_{X,Y}$, then

$$\mathbb{E}[g(X, Y)] = \int_{\mathbb{R}^2} g(x, y) f_{X,Y}(x, y)\, dx\, dy$$

for all measurable functions g such that $g(X, Y) \in L^1(\Omega)$.

(vii) Jensen's inequality: If $f : \mathbb{R} \to \mathbb{R}$ is measurable and convex, then $f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)]$.

The **covariance** and **variance** of two random variables is defined as

$$\mathrm{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])], \quad \mathrm{Var}[X] = \mathrm{Cov}(X, X).$$

The set of random variables such that $\mathbb{E}[X^2]$ is finite will be denoted by $L^2(\Omega)$. By Jensen inequality with $f(x) = x^2$, $L^2(\Omega) \subset L^1(\Omega)$.

Using (v) in Theorem 1 one can prove that

$$X \in \mathcal{N}(m, \sigma^2) \Rightarrow \mathrm{Var}[X] = \sigma^2, \quad X \in \chi^2(\delta, \beta) \Rightarrow \mathrm{Var}[X] = 2(\delta + 2\beta).$$

Moreover if X_1, X_2 are jointly normally distributed with covariant matrix C, then $C_{ij} = \mathrm{Cov}(X_i, X_j)$, $i, j = 1, 2$.

Characteristic function

Let X be a random variable with finite expectation. The function $\theta_X : \mathbb{R} \to \mathbb{C}$ given by

$$\theta_X(u) = \mathbb{E}[e^{iuX}]$$

is called the **characteristic function of X.**

If the random variable X admits the density f_X, then

$$\theta_X(u) = \int_{\mathbb{R}} e^{iux} f_X(x)\, dx,$$

i.e., the characteristic function is the inverse Fourier transform of the density. For example:

$$X \in \mathcal{N}(m, \sigma^2) \Rightarrow \theta_X(u) = \exp(ium - \frac{1}{2}\sigma^2 u^2), \tag{2}$$

$$X \in \chi^2(\delta, \beta) \Rightarrow \theta_X(u) = (1 - 2iu)^{-\delta/2} \exp\left(-\frac{\beta u}{2u + i}\right) \tag{3}$$

It can be shown that $\theta_X = \theta_Y$ if and only if $F_X = F_Y$. In particular, if one wants to show that $X \in \chi^2(\delta, \beta)$, it suffices to show that its characteristic function is given as in (3).

2 Stochastic processes. Brownian motion

A **stochastic process** is a one-parameter family of random variables, which we denote by $\{X(t)\}_{t\geq 0}$, or by $\{X(t)\}_{t\in[0,T]}$ if the parameter t is restricted to the interval $[0, T]$, $T > 0$.

In most applications t is the time variable.

Hence, for each time $t \geq 0$, $X(t) : \Omega \to \mathbb{R}$ is a random variable.

We denote by $X(t, \omega)$ the value of $X(t)$ on the sample point $\omega \in \Omega$, i.e., $X(t, \omega) = X(t)(\omega)$.

For each $\omega \in \Omega$ fixed, the curve $\gamma_X^\omega : \mathbb{R} \to \mathbb{R}$, $\gamma_X^\omega(t) = X(t, \omega)$ is called the **ω-path** of the stochastic process and is assumed to be a measurable function.

If the paths of a stochastic process are all equal, we say that the stochastic process is a **deterministic function of time**.

Two stochastic processes $\{X(t)\}_{t\geq 0}$, $\{Y(t)\}_{t\geq 0}$ are said to be independent if for all $m, n \in \mathbb{N}$ and $0 \leq t_1 < t_2 < \cdots < t_n$, $0 \leq s_1 < s_2 < \cdots < s_m$, the σ-algebras $\sigma(X(t_1), \dots, X(t_n))$, $\sigma(Y(s_1), \dots, Y(s_m))$ are independent.

The filtration generated by the stochastic process $\{X(t)\}_{t\geq 0}$ is given by $\{\mathcal{F}_X(t)\}_{t\geq 0}$, where

$$\mathcal{F}_X(t) = \mathcal{F}_{\mathcal{O}(t)}, \quad \mathcal{O}(t) = \cup_{0\leq s\leq t}\sigma(X(s)).$$

If $\{\mathcal{F}(t)\}_{t\geq 0}$ is a filtration and $\mathcal{F}_X(t) \subseteq \mathcal{F}(t)$, for all $t \geq 0$, we say that the stochastic process $\{X(t)\}_{t\geq 0}$ is **adapted** to the filtration $\{\mathcal{F}(t)\}_{t\geq 0}$.

A **Brownian motion** (or **Wiener process**) is a stochastic process $\{W(t)\}_{t\geq 0}$ such that

(i) The paths are continuous and start from 0 almost surely, i.e., the sample points $\omega \in \Omega$ such that $\gamma_W^\omega(0) = 0$ and γ_W^ω is a continuous function comprise a set of probability 1;

(ii) The increments over disjoint time intervals are independent, i.e., for all $0 = t_0 < t_1 < \cdots < t_m \in (0, \infty)$, the random variables

$$W(t_1) - W(t_0),\ W(t_2) - W(t_1),\ \dots,\ W(t_m) - W(t_{m-1})$$

are independent;

(iii) For all $s < t$, the increment $W(t) - W(s)$ belongs to $\mathcal{N}(0, t - s)$.

The properties defining a Brownian motion depend on the probability measure $\mathbb{P}$.

Thus a stochastic process may be a Brownian motion relative to a probability measure $\mathbb{P}$ and not a Brownian motion with respect to another (possibly equivalent) probability measure $\widetilde{\mathbb{P}}$.

If we want to emphasize the probability measure $\mathbb{P}$ with respect to which a stochastic process is a Brownian motion we shall say that it is a $\mathbb{P}$-Brownian motion.

Let $\{W(t)\}_{t\geq 0}$ be a Brownian motion and denote by $\sigma^+(W(t))$ the σ-algebra generated by the increments $\{W(s) - W(t); s \geq t\}$, that is

$$\sigma^+(W(t)) = \mathcal{F}_{O(t)},\ \mathcal{O}(t) = \cup_{s\geq t}\sigma(W(s) - W(t)).$$

A filtration $\{\mathcal{F}(t)\}_{t\geq 0}$ is said to be a **non-anticipating** filtration for the Brownian motion $\{W(t)\}_{t\geq 0}$ if $\{W(t)\}_{t\geq 0}$ is adapted to $\{\mathcal{F}(t)\}_{t\geq 0}$ and if the σ-algebras $\sigma^+(W(t))$, $\mathcal{F}(t)$ are independent for all $t \geq 0$.

3 Quadratic variation

Let $n \in \mathbb{N}$ and $\Pi_n = \{t_0 = 0, t_1^{(n)}, t_2^{(n)}, \dots, t_{m(n)-1}^{(n)}, t_{m(n)} = T\}$ be a partition of the interval $[0, T]$.

Hence $\{\Pi_n\}_{n\in\mathbb{N}}$ is sequence of partitions of the interval $[0, T]$.

Assume that the size of the partitions in the sequence goes to zero as $n \to \infty$, i.e.,

$$\|\Pi_n\| \to 0 \text{ as } n \to \infty, \text{ where } \|\Pi_n\| = \max_j (t_{j+1}^{(n)} - t_j^{(n)}).$$

We say that the stochastic process $\{X(t)\}_{t\geq 0}$ has L^2**-quadratic variation** $[X, X](T)$ in the interval $[0, T]$ along the sequence of partitions $\{\Pi_n\}_{n\in\mathbb{N}}$ if

$$\lim_{n\to\infty} \mathbb{E}\left[\left(\sum_{j=0}^{m(n)-1} (X(t_{j+1}^{(n)}) - X(t_j^{(n)}))^2 - [X, X](T)\right)^2\right] = 0.$$

Note that $[X, X](T)$ is a (non-negative) random variable.

If there exists a process $\{q(t)\}_{t\geq 0}$ such that

$$[X, X](T) = \int_0^T q(t)\, dt, \quad \textit{along any sequence of partitions}$$

then we write

$$dX(t)dX(t) = q(t)\, dt.$$

The process $\{q(t)\}_{t\geq 0}$ is called **rate of quadratic variation** and measures how fast quadratic variation accumulates in time in the stochastic process $\{X(t)\}_{t\geq 0}$.

For example, it can be show that

$$dW(t)dW(t) = dt, \quad dt\, dt = 0. \tag{4}$$

Similarly, we say that two stochastic processes $\{X_1(t)\}_{t\geq 0}$ and $\{X_2(t)\}_{t\geq 0}$ have L^2-**cross variation** $[X_1, X_2](T)$ in the interval $[0,T]$ along the sequence of partitions $\{\Pi_n\}_{n\in\mathbb{N}}$, if

$$\lim_{n\to\infty} \mathbb{E}\left[\left(\sum_{j=0}^{m(n)-1} (X_1(t_{j+1}^{(n)}) - X_1(t_j^{(n)}))(X_2(t_{j+1}^{(n)}) - X_2(t_j^{(n)})) - [X_1, X_2](T)\right)^2\right] = 0,$$

and write

$$dX_1(t)dX_2(t) = \xi(t)\,dt$$

if

$$[X_1, X_2](T) = \int_0^T \xi(t)\,dt \quad \textit{along any sequence of partitions}$$

For instance it can be proved that

$$dW(t)dt = 0. \tag{5}$$

Moreover if $\{W_1(t)\}_{t\geq 0}$, $\{W_2(t)\}_{t\geq 0}$ are two independent Brownian motions then

$$dW_1(t)dW_2(t) = 0. \tag{6}$$