Lecture 5

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Financial derivatives and PDE's
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$\chi^2(\delta, \beta)$

Exercise 2.8

Let $X \in \mathcal{N}(0,1)$ and $Y = X^2$. Show that $Y \in \chi^2(1)$.

SOLUTION:

$$\chi^2(\delta) = \underbrace{\chi^2(\delta, 0)}$$

$$X \in \chi^2(\delta) \Rightarrow f_X(x) = \frac{x^{\delta/2 - 1} e^{-x/2}}{2^{\delta/2}\,\Gamma(\delta/2)} \mathbb{I}_{\underline{x>0}} \quad \left(\frac{e^{-x/2}}{\sqrt{x}\,\sqrt{2\pi}} \text{ FOR } \delta = 1 \right)$$

$$\Gamma(t) = \int_0^\infty z^{t-1} e^{-z}\, dz \qquad \left(\Gamma(1/2) = \sqrt{\pi} \right)$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} \mathbb{P}(Y \le y) = \frac{d}{dy} \mathbb{P}\underbrace{(X^2 \le y)}$$

$\boxed{y \geq 0}$

$$\stackrel{!}{=} \frac{d}{dy} \mathbb{P}(-\sqrt{y} \le X \le \sqrt{y}) = \frac{d}{dy} \int_{-\sqrt{y}}^{\sqrt{y}} e^{-\frac{1}{2}z^2} \frac{dz}{\sqrt{2\pi}}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left(e^{-\frac{1}{2}z^2}\right)_{z=\sqrt{y}} \frac{d}{dy}\sqrt{y} - \left(e^{-\frac{1}{2}z^2}\right)_{z=-\sqrt{y}} \frac{d}{dy}(-\sqrt{y}) \right] = \frac{1}{\sqrt{2\pi}} \frac{e^{-y/2}}{\sqrt{y}}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\left(e^{-\frac{1}{2}z^2}\right)_{z=\sqrt{y}} \frac{d}{dy}\sqrt{y} - \left(e^{-\frac{1}{2}z^2}\right)_{z=-\sqrt{y}} \frac{d}{dy}(-\sqrt{y})\right] = \frac{1}{\sqrt{2\pi}} \frac{e^{-y/2}}{\sqrt{y}}$$

$X : \Omega \to [0, \infty)$

$\mathbb{P}(X = 0) = 0\ ?$ $\lim_{\varepsilon \to 0} \mathbb{P}(0 \le X < \varepsilon) = 0$

Exercise 2.15

Derive the density of the geometric Brownian motion and use the result to show that $\mathbb{P}(S(t) = 0) = 0$, i.e., a stock whose price is described by a geometric Brownian motion cannot default.

SOLUTION:

$$S(t) = S(0)\, e^{\alpha t + \sigma W(t)}$$ GEOMETRIC BM

$$f_{S(t)}(x) = \frac{d}{dx} F_{S(t)}(x) = \frac{d}{dx} \mathbb{P}(S(t) \le x)$$

(THIS IS EQUAL TO ZERO IF $x \le 0$)

FOR $x > 0$ WE WRITE

$$S(t) \le x \iff S(0)\, e^{\alpha t + \sigma W(t)} \le x$$

$$\iff W(t) \le \left(\log \frac{x}{S(0)} - \alpha t\right)/\sigma = A(x)$$

$$\mathbb{P}(S(t) \le x) = \mathbb{P}(W(t) \le A(x)) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{A(x)} e^{-\frac{y^2}{2t}}\, dy$$

$$f_{S(t)}(x) = \frac{d}{dx} \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{A(x)} e^{-\frac{y^2}{2t}}\, dy = \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{A(x)^2}{2t}} \frac{d}{dx} A(x)$$

$$= \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{A(x)^2}{2t}} \frac{1}{\sigma x} \implies f_{S(t)}(x) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \frac{e^{-\frac{(\log x - \log S(0) - \alpha t)^2}{2\sigma^2 t}}}{x} \mathbb{1}_{x>0}$$

(LOG-NORMAL DISTRIBUTION)

SINCE $\int_0^\infty f_{S(t)}(x)\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}\, dy = 1$

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$$\int f g\, dx \le \left(\int f^2 dx\right)^{1/2} \left(\int g^2 dx\right)^{1/2}$$

$$\int f g \, dx \leq \left(\int f^2 dx\right)^{1/2} \left(\int g^2 dx\right)^{1/2}$$

Exercise 3.3

Prove the Schwarz inequality,

$$\mathbb{E}[XY] \leq \sqrt{\mathbb{E}[X^2]\mathbb{E}[Y^2]}, \qquad (1)$$

for all random variables $X, Y \in L^2(\Omega)$.

SOLUTION:

FIRST, THIS IS OBVIOUS IF $Y = 0$ (AS) ASSUME THAT Y IS NOT IDENTICALLY ZERO. EQUIVALENTLY, ASSUME $\mathbb{E}[Y^2] > 0$. DEFINE

$$Z = X - \frac{\mathbb{E}[XY]}{\mathbb{E}[Y^2]} Y$$

$$0 \leq \mathbb{E}[Z^2] = \mathbb{E}[X^2] + \frac{\mathbb{E}[XY]^2}{\mathbb{E}[Y^2]^{\cancel{2}}} \cancel{\mathbb{E}[Y^2]}$$

$$- 2\frac{\mathbb{E}[XY]}{\mathbb{E}[Y^2]} \mathbb{E}[XY] = \frac{\mathbb{E}[X^2]\mathbb{E}[Y^2] - \mathbb{E}[XY]^2}{\mathbb{E}[Y^2]}$$

$$\Rightarrow \mathbb{E}[X^2]\mathbb{E}[Y^2] \geq \mathbb{E}[XY]^2 \Leftrightarrow (1)$$

Exercise 3.27

The purpose of this exercise is to show that the conditional expectation is the best estimator

The purpose of this exercise is to show that the conditional expectation is the best estimator of a random variable when some information is given in the form of a sub-σ-algebra. Let $X \in L^1(\Omega)$ and $\mathcal{G} \subseteq \mathcal{F}$ be a sub-σ-algebra. Define $\mathrm{Err} = X - \mathbb{E}[X|\mathcal{G}]$. Show that $\mathbb{E}[\mathrm{Err}] = 0$, and

$$\mathrm{VAR}[\mathbb{E}[X|\mathcal{G}]-X] = \mathbb{E}[(\mathrm{Err})^2] = \mathrm{Var}[\mathrm{Err}] = \min_Y \mathrm{Var}[Y - X]$$

THE MINIMIZER IS THEREFORE $Y = \mathbb{E}[X|\mathcal{G}]$

where the minimum is taken with respect to all $\mathcal{G}$-measurable random variables Y.

SOLUTION

$$\mathbb{E}[\mathrm{Err}] = \mathbb{E}[X - \mathbb{E}[X|\mathcal{G}]] = \mathbb{E}[X] - \mathbb{E}[\mathbb{E}[X|\mathcal{G}]]$$

$$= \mathbb{E}[X] - \mathbb{E}[X] = 0$$

$$\mu = \mathbb{E}[Y-X]$$

$$\mathrm{VAR}[Y-X] = \mathbb{E}[(Y-X-\mu)^2] = \mathbb{E}[(Y-X-\mu+\mathbb{E}[X|\mathcal{G}]-\mathbb{E}[X|\mathcal{G}])^2]$$

$$= \mathbb{E}[(\overbrace{\mathbb{E}[X|\mathcal{G}]-X}^{\mathrm{Err}})^2 + (Y-\mu-\mathbb{E}[X|\mathcal{G}])^2$$

$$+ 2(\mathbb{E}[X|\mathcal{G}]-X)(Y-\mu-\mathbb{E}[X|\mathcal{G}])]$$

$$= \mathrm{VAR}[\mathrm{Err}] + \mathbb{E}[(Y-\mu-\mathbb{E}[X|\mathcal{G}])^2]$$

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$$\mathbb{E}[\beta] = \mathbb{E}[\mathbb{E}[\beta|\mathcal{G}]] = \mathbb{E}[(Y-\mu-\mathbb{E}[X|\mathcal{G}])\,\mathbb{E}[(\mathbb{E}[X|\mathcal{G}]-X)|\mathcal{G}]] = 0$$

$$\mathbb{E}[X|\mathcal{G}] - \mathbb{E}[X|\mathcal{G}] = 0$$

HENCE $\mathrm{VAR}[Y-X] \geq \mathrm{VAR}[\mathrm{Err}]$. ON THE OTHER HAND

$\mathrm{VAR}[\mathrm{Err}] = \mathrm{VAR}[Y-X]\big|_{Y=\mathbb{E}[X|\mathcal{G}]} \Rightarrow Y = \mathbb{E}[X|\mathcal{G}]$ MINIMIZES $\mathrm{VAR}[Y-X]$ IN THE SPACE OF $\mathcal{G}$-MEAS. R.V.'S.

Exercise 4.4

Let $\{W_1(t)\}_{t\geq 0}$, $\{W_2(t)\}_{t\geq 0}$ be Brownian motions. Assume that there exists a constant $\rho \in [-1, 1]$ such that $dW_1(t)dW_2(t) = \rho dt$. Show that ρ is the correlation of the two Brownian motions at time t. [Assuming that $\{W_1(t)\}_{t\geq 0}$, $\{W_2(t)\}_{t\geq 0}$ are independent, compute $\mathbb{P}(W_1(t) > W_2(s))$, for all $s, t > 0$.] DO IT YOURSELF

SOLUTION:

SOLUTION:

$$dW_1(t)\,dW_2(t) = \rho\, dt \iff [W_1, W_2](T) = \rho T$$

ALONG ANY SEQUENCE OF PARTITIONS

$$\mathrm{CORR}[W_1(t), W_2(t)] = \frac{\mathrm{Cov}(W_1(t), W_2(t))}{\sqrt{\underbrace{\mathrm{VAR}[W_1(t)]}_{t}\,\underbrace{\mathrm{VAR}[W_2(t)]}_{t}}} = \frac{\mathbb{E}[W_1(t)W_2(t)]}{t}$$

$$= \rho \iff \mathbb{E}[W_1(t)W_2(t)] = \rho t$$

$$d(W_1(t)W_2(t)) = (dW_1(t))\,W_2(t) + W_1(t)\,dW_2(t) + \underbrace{dW_1(t)\,dW_2(t)}_{\substack{\parallel \\ \rho\, dt \text{ BY ASSUMPTION}}}$$

THIS MEANS

$$W_1(t)W_2(t) = \int_0^t W_2(s)\,dW_1(s) + \int_0^t W_1(s)\,dW_2(s) + \rho t$$

$$\Rightarrow \mathbb{E}[W_1(t)W_2(t)] = \mathbb{E}\Big[\underbrace{\int_0^t W_2(s)\,dW_1(s)}\Big] + \mathbb{E}\Big[\underbrace{\int_0^t W_1(s)\,dW_2(s)}\Big] + \rho t$$

MARTINGALES

HENCE $\mathbb{E}\left[\int_0^t W_2(s)\,dW_1(s)\right] = \mathbb{E}\left[\int_0^t W_1(s)\,dW_2(s)\right] = 0$

$$\Rightarrow \mathbb{E}[W_1(t)W_2(t)] = \rho t \qquad \blacksquare$$

Exercise 4.5

Consider the stochastic process $\{X(t)\}_{t\geq 0}$ defined by $X(t) = W(t)^3 - 3tW(t)$. Show that $\{X(t)\}_{t\geq 0}$ is a martingale and find a process $\{\Gamma(t)\}_{t\geq 0}$ adapted to $\{\mathcal{F}_W(t)\}_{t\geq 0}$ such that

$$(*) \longrightarrow \quad X(t) = X(0) + \int_0^t \Gamma(s)dW(s).$$

(The existence of the process $\{\Gamma(t)\}_{t\geq 0}$ is ensured by Theorem 4.5(v).)

MARTINGALE REPRESENTATION THEOREM.

SOLUTION: $(*)$ IS, IN THE STOCH. DIFF. NOTATION,

$$dX(t) = \Gamma(t)\,dW(t)$$

IN FACT

$$dX(t) = \cancel{-3W(t)\,dt} + (3W(t)^2 - 3t)\,dW(t)$$

$$\cancel{+\, 3W(t)\, \underbrace{dW(t)\,dW(t)}_{dt}} = \Gamma(t)\,dW(t)$$

WHERE $\Gamma(t) = 3W(t)^2 - 3t \in L^2[\mathcal{F}_W(t)]$

$$X(t) = f(t, W(t))$$

$$f(t,x) = x^3 - 3tx$$

Exercise 4.6

Let $\{\theta(t)\}_{t\geq 0} \in \mathcal{C}^0[\mathcal{F}(t)]$ and define the stochastic process $\{Z(t)\}_{t\geq 0}$ by

$$Z(t) = \exp\left(-\int_0^t \theta(s)dW(s) - \frac{1}{2}\int_0^t \theta^2(s)ds\right).$$

Show that

$$Z(t) = 1 - \int_0^t \theta(s)Z(s)\, dW(s).$$

SOLUTION

$$Z(t) = e^{X(t)} \qquad X(t) = -\int_0^t \theta(s)\, dW(s) - \frac{1}{2}\int_0^t \theta(s)^2\, ds$$

$$\Rightarrow \quad dX(t) = -\theta(t)\, dW(t) - \frac{1}{2}\theta(t)^2\, dt$$

HENCE, BY ITÔ'S FORMULA,

$$\theta(t)\, dW(t)\, \theta(t)\, dW(t) = \theta(t)^2\, dt$$

$$dZ(t) = e^{X(t)}\, dX(t) + \frac{1}{2} e^{X(t)}\, dX(t)\, dX(t)$$

$$= Z(t)\left[-\theta(t)\, dW(t) - \cancel{\frac{1}{2}\theta^2(t)\, dt} + \cancel{\frac{1}{2}\theta(t)^2\, dt}\right]$$

$$= -Z(t)\,\theta(t)\, dW(t)$$

$$Z(t) = Z(0) - \int_0^t Z(s)\,\theta(s)\, dW(s) =$$

$$= 1 - \int_0^t Z(s)\,\theta(s)\, dW(s)$$