Financial derivatives and PDE's
Lecture 9

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The CIR process

A **CIR process** is a stochastic process $\{X(t)\}_{t\geq s}$ satisfying the SDE

$$dX(t) = a(b - X(t))\,dt + c\sqrt{X(t)}\,dW(t), \quad X(s) = x > 0, \tag{1}$$

where a, b, c are constant ($c \neq 0$).

CIR processes are used in finance to model the stock volatility in the Heston model and the spot interest rate of bonds in the CIR model.

Note that the SDE (1) is not of the form considered so far, as the function $\beta(t,x) = c\sqrt{x}$ is defined only for $x \geq 0$ and, more importantly, it is not Lipschitz continuous in a neighborhood of $x = 0$. Nevertheless it can be shown that (1) admits a unique global solution for all $x > 0$.

Clearly the solution satisfies $X(t) \geq 0$ a.s., for all $t \geq 0$, otherwise the Itô integral in the right hand side of (1) would not even be defined.

For future applications, it is important to know whether the solution can hit zero in finite time with positive probability. This question is answered in the following theorem.

Theorem 1. *Let $\{X(t)\}_{t\geq 0}$ be the CIR process with initial value $X(0) = x > 0$ at time $t = 0$. Define*

$$\tau_0^x = \inf\{t \geq 0 : X(t) = 0\}.$$

Then $\mathbb{P}(\tau_0^x < \infty) = 0$ if and only if $ab \geq c^2/2$, which is called **Feller's condition**.

The following theorem shows how to build a CIR process from a family of linear SDE's.

Theorem 2. *Let $\{W_1(t)\}_{t\geq 0}, \dots \{W_N(t)\}_{t\geq 0}$ be $N \geq 2$ independent Brownian motions and assume that $\{X_1(t)\}_{t\geq 0}, \dots, \{X_N(t)\}_{t\geq 0}$ solve*

$$dX_j(t) = -\frac{\theta}{2} X_j(t)\, dt + \frac{\sigma}{2}\, dW_j(t), \quad j = 1, \dots, N, \quad X_j(0) = x_j \in \mathbb{R}, \tag{2}$$

where θ, σ are deterministic constant. There exists a Brownian motion $\{W(t)\}_{t\geq 0}$ such that the stochastic process $\{X(t)\}_{t\geq 0}$ given by

$$X(t) = \sum_{j=1}^{N} X_j(t)^2$$

solves (1) *with $a = \theta$, $c = \sigma$ and $b = \frac{N\sigma^2}{4\theta}$.*

Proof. Let $X(t) = \sum_{j=1}^{N} X_j(t)^2$. Applying Itô's formula we find, after straightforward calculations,

$$dX(t) = (\frac{N\sigma^2}{4} - \theta X(t))\, dt + \sigma \sum_{j=1}^{N} X_j(t)\, dW_j(t).$$

Letting $a = \theta$, $c = \sigma$, $b = \frac{N\sigma^2}{4\theta}$ and

$$dW(t) = \sum_{j=1}^{N} \frac{X_j(t)}{\sqrt{X(t)}}\, dW_j(t),$$

we obtain that $X(t)$ satisfies

$$dX(t) = a(b - X(t))\, dt + c\sqrt{X(t)}\, dW(t).$$

Thus $\{X(t)\}_{t\geq 0}$ is a CIR process, provided we prove that $\{W(t)\}_{t\geq 0}$ is a Brownian motion. We shall use the so-called **Lévy characterization theorem**:

Let $\{M(t)\}_{t\geq 0}$ be a martingale relative to a filtration $\{\mathcal{F}(t)\}_{t\geq 0}$. Assume that (i) $M(0) = 0$ a.s., (ii) the paths $t \to M(t, \omega)$ are a.s. continuous and (iii) $dM(t)dM(t) = dt$. Then $\{M(t)\}_{t\geq 0}$ is a Brownian motion and $\{\mathcal{F}(t)\}_{t\geq 0}$ a non-anticipating filtration thereof.

Clearly, $W(0) = 0$ a.s. and the paths $t \to W(t, \omega)$ are a.s. continuous. Moreover $\{W(t)\}_{t\geq 0}$ is a martingale, as it is the sum of martingale Itô integrals. Hence to conclude that $\{W(t)\}_{t\geq 0}$ is a Brownian motion we must show that $dW(t)dW(t) = dt$. We have

$$
\begin{aligned}
dW(t)dW(t) &= \frac{1}{X(t)} \sum_{i,j=1}^{N} X_i(t)X_j(t)dW_i(t)dW_j(t) = \frac{1}{X(t)} \sum_{i,j=1}^{N} X_i(t)X_j(t)\delta_{ij}dt \\
&= \frac{1}{X(t)} \sum_{j=1}^{N} X_j^2(t)dt = dt,
\end{aligned}
$$

where we used that $dW_i(t)dW_j(t) = \delta_{ij}dt$, since the Brownian motions are independent. $\square$

Remark:

The process $\{X(t)\}_{t\geq 0}$ in Theorem 2 is said to be a **weak solution** of (1), because the Brownian motion $\{W(t)\}_{t\geq 0}$ in the SDE is not given advance, but rather it depends on the solution itself

Since $N \geq 2$ in the previous theorem implies the Feller condition $ab \geq c^2/2$, then (provided $x_j \neq 0$ for some j, so that $X(0) > 0$) the CIR process constructed in Theorem 2 does not hit zero, see Theorem 1.

Moreover, since the solution of (2) is

$$
X_j(t) = e^{-\frac{1}{2}\theta t} \left(x_j + \frac{\sigma}{2} \int_0^t e^{\frac{1}{2}\theta\tau} dW_j(\tau) \right),
$$

then the random variables $X_1(t), \ldots, X_N(t)$ are normally distributed with

$$
\mathbb{E}[X_j(t)] = e^{-\frac{1}{2}\theta t} x_j, \quad \mathrm{Var}[X_j(t)] = \frac{\sigma^2}{4\theta}(e^{\frac{1}{2}\theta t} - 1).
$$

It follows that the CIR process constructed Theorem 2 is non-central χ^2 distributed.

The following theorem shows that this is a general property of CIR processes.

Theorem 3. *Assume $ab > 0$. The CIR process starting at $x > 0$ at time $t = s$ satisfies*

$$X(t; s, x) = \frac{1}{2k} Y, \quad Y(t; s, x) \in \chi^2(\delta, \beta),$$

where

$$k = \frac{2a}{(1 - e^{-a(t-s)})c^2}, \quad \delta = \frac{4ab}{c^2}, \quad \beta = 2kxe^{-a(t-s)}.$$

Proof. As the CIR process is a time-homogeneous Markov process, it is enough to prove the claim for $s = 0$.

Let $X(t) = X(t; 0, x)$ for short. The characteristic function of $X(t)$ is given by

$$\theta_{X(t)}(u) = \mathbb{E}[e^{iuX(t)}] = \mathbb{E}[e^{iu\frac{Y(t)}{2k}}] = \theta_{Y(t)}(\frac{u}{2k})$$

where $Y(t) = Y(t, 0, x)$. Thus the statement of the theorem is equivalent to

$$h(t, u) := \theta_{X(t)}(u) = \frac{\exp\Big(-\frac{\beta u}{2(u+ik)}\Big)}{(1 - iu/k)^{\delta/2}}, \tag{3}$$

where $k = \frac{2a}{(1-e^{-at})c^2}$, $\delta = \frac{4ab}{c^2}$, $\beta = 2kxe^{-at}$.

To prove this denote $p(t, 0, x, y) = p_*(t, x, y)$ the transition density of $X(t)$. Then

$$h(t, u) = \int_{\mathbb{R}} e^{iuy} p_*(t, x, y)\, dy. \tag{4}$$

p_* solves the Fokker-Planck equation

$$\partial_t p_* + \partial_y (a(b - y)p_*) - \frac{1}{2}\partial_y^2 (c^2 y\, p_*) = 0, \tag{5}$$

with initial datum $p_*(0, x, y) = \delta(x - y)$.

After straightforward calculations we derive the following equation on h

$$\partial_t h - iabuh + (au - \frac{c^2}{2} iu^2)\partial_u h = 0. \tag{6a}$$

The initial condition for equation (6a) is

$$h(0, u) = e^{iux}, \tag{6b}$$

which is equivalent to $p_*(0, x, y) = \delta(x - y)$.

Now the proof can be completed by showing that (3) satisfies the initial value problem (6a)-(6b).

□

By Theorem 7 , for $ab > 0$, the density of the CIR process starting at x is $f_{\mathrm{CIR}}(y; t - s, x)$, where

$$f_{\mathrm{CIR}}(y; \tau, x) = k e^{aq\tau/2} \exp(-k(y + xe^{-a\tau})) \left(\frac{y}{x}\right)^{q/2} I_q(2ke^{-a\tau/2}\sqrt{xy}), \quad q = \frac{\delta}{2} - 1. \tag{7}$$

Finally we discuss briefly the question of existence of strong solutions to the Kolmogorov equation for the CIR process, which is

$$\partial_t u + a(b - x)\partial_x u + \frac{c^2}{2} x \partial_x^2 u = 0, \quad (t, x) \in \mathcal{D}_T^+, \quad u(T, x) = g(x). \tag{8}$$

Note carefully that the Kolmogorov PDE is now defined only for $x > 0$, as the initial value x in (1) must be positive.

Now, if a strong solution of (8) exists, then it must be given by $u(t, x) = \mathbb{E}[g(X(T; t, x))]$.

Supposing $ab > 0$, then

$$u(t, x) = \mathbb{E}[g(X(T; t, x))] = \int_0^\infty f_{\mathrm{CIR}}(y; T - t, x) g(y)\, dy,$$

where $f_{\mathrm{CIR}}(y; \tau, x)$ is given by (7). Using the asymptotic behavior of $f_{\mathrm{CIR}}(y; \tau, x)$ as $x \to 0^+$, it can be shown $\partial_x u(t, x)$ is bounded near the axis $x = 0$ only if the Feller condition $ab \geq c^2/2$ is satisfied.

Hence u is the (unique) strong solution of (8) if and only if $ab \geq c^2/2$.