Financial derivatives and PDE's
Lecture 10

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1 Radon-Nikodým theorem

We shall need the following characterization of equivalent probability measures.

Theorem 1. *Given a probability measure $\mathbb{P}$, the following are equivalent:*

(i) $\widetilde{\mathbb{P}}$ *is a probability measure equivalent to* $\mathbb{P}$*;*

(ii) There exists a random variable $Z : \Omega \to \mathbb{R}$ *such that* $Z > 0$ $\mathbb{P}$*-almost surely,* $\mathbb{E}[Z] = 1$ *and* $\widetilde{\mathbb{P}}(A) = \mathbb{E}[Z\mathbb{I}_A]$*, for all* $A \in \mathcal{F}$*.*

Moreover, assuming any of these two equivalent conditions, the random variable Z is unique (up to a $\mathbb{P}$-null set) and for all random variables X such that $XZ \in L^1(\Omega, \mathbb{P})$, we have $X \in L^1(\Omega, \widetilde{\mathbb{P}})$ and

$$\widetilde{\mathbb{E}}[X] = \mathbb{E}[ZX]. \tag{1}$$

Now assume that $\{Z(t)\}_{t\geq 0}$ is a martingale such that $Z(t) > 0$ a.s. and $\mathbb{E}[Z(0)] = 1$.

Since martingales have constant expectation, then $\mathbb{E}[Z(t)] = 1$ for all $t \geq 0$.

By Theorem 1, the map $\widetilde{\mathbb{P}} : \mathcal{F} \to [0, 1]$ given by

$$\widetilde{\mathbb{P}}(A) = \mathbb{E}[Z(T)\mathbb{I}_A], \quad A \in \mathcal{F} \tag{2}$$

is a probability measure equivalent to $\mathbb{P}$, for all $T > 0$.

Note that $\widetilde{\mathbb{P}}$ depends on $T > 0$ and $\widetilde{\mathbb{P}} = \mathbb{P}$, for $T = 0$. The dependence on T is however not reflected in our notation.

As usual, the (conditional) expectation in the probability measure $\widetilde{\mathbb{P}}$ will be denoted $\widetilde{\mathbb{E}}$.

The relation between $\mathbb{E}$ and $\widetilde{\mathbb{E}}$ is revealed in the following theorem.

Theorem 2. *Let $\{Z(t)\}_{t\geq 0}$ be a $\mathbb{P}$-martingale relative to a filtration $\{\mathcal{F}(t)\}_{t\geq 0}$ such that $Z(t) > 0$ a.s. and $\mathbb{E}[Z(0)] = 1$. Let $T > 0$ and let $\widetilde{\mathbb{P}}$ be the probability measure equivalent to $\mathbb{P}$ defined by (2). Let $t \in [0, T]$ and let X be a $\mathcal{F}(t)$-measurable random variable such that $Z(t)X \in L^1(\Omega, \mathbb{P})$. Then $X \in L^1(\Omega, \widetilde{\mathbb{P}})$ and*

$$\widetilde{\mathbb{E}}[X] = \mathbb{E}[Z(t)X]. \tag{3}$$

Moreover, for all $0 \leq s \leq t$ and for all random variables Y such that $Z(t)Y \in L^1(\Omega, \mathbb{P})$, there holds

$$\widetilde{\mathbb{E}}[Y|\mathcal{F}(s)] = \frac{1}{Z(s)}\mathbb{E}[Z(t)Y|\mathcal{F}(s)]. \tag{4}$$

2 Girsanov's theorem

In this section we assume that the non-anticipating filtration of the Brownian motion coincides with $\{\mathcal{F}_W(t)\}_{t\geq 0}$.

Let $\{\theta(t)\}_{t\geq 0} \in \mathcal{C}^0[\mathcal{F}_W(t)]$ satisfy the Novikov condition. It follows that the positive stochastic process $\{Z(t)\}_{t\geq 0}$ given by

$$Z(t) = \exp\left(-\int_0^t \theta(s)dW(s) - \frac{1}{2}\int_0^t \theta^2(s)ds\right) \tag{5}$$

is a martingale relative to $\{\mathcal{F}_W(t)\}_{t\geq 0}$.

As $Z(0) = 1$, then $\mathbb{E}[Z(t)] = 1$ for all $t \geq 0$. Thus we can use the stochastic process $\{Z(t)\}_{t\geq 0}$ to generate a measure $\widetilde{\mathbb{P}}$ equivalent to $\mathbb{P}$, namely $\widetilde{\mathbb{P}} : \mathcal{F} \to [0, 1]$ is given by

$$\widetilde{\mathbb{P}}(A) = \mathbb{E}[Z(T)\mathbb{I}_A], \quad A \in \mathcal{F}, \tag{6}$$

for some given $T > 0$, see Theorem 2. The relation between $\mathbb{E}$ and $\widetilde{\mathbb{E}}$ is

$$\widetilde{\mathbb{E}}[X] = \mathbb{E}[Z(t)X], \tag{7}$$

for all $t \geq 0$ and $\mathcal{F}_W(t)$-measurable random variables X, and

$$\widetilde{\mathbb{E}}[Y|\mathcal{F}_W(s)] = \frac{1}{Z(s)}\mathbb{E}[Z(t)Y|\mathcal{F}_W(s)] \tag{8}$$

for all $0 \leq s \leq t$ and random variables Y.

We can now state and sketch the proof of **Girsanov**'s theorem, which is a fundamental result with deep applications in mathematical finance.

Theorem 3. *Define the stochastic process* $\{\widetilde{W}(t)\}_{t\geq 0}$ *by*

$$\widetilde{W}(t) = W(t) + \int_0^t \theta(s)ds, \tag{9}$$

i.e., $d\widetilde{W}(t) = dW(t) + \theta(t)dt$. *Then* $\{\widetilde{W}(t)\}_{t\geq 0}$ *is a* $\widetilde{\mathbb{P}}$*-Brownian motion with non-anticipating filtration* $\{\mathcal{F}_W(t)\}_{t\geq 0}$.

Sketch of the proof. We prove only that $\{\widetilde{W}(t)\}_{t\geq 0}$ is a $\widetilde{\mathbb{P}}$-Brownian motion using the Lévy characterization of Brownian motions. Clearly, $\{\widetilde{W}(t)\}_{t\geq 0}$ starts from zero and has continuous paths a.s. Moreover we (formally) have $d\widetilde{W}(t)d\widetilde{W}(t) = dW(t)dW(t) = dt$. Hence it remains to show that the Brownian motion $\{\widetilde{W}(t)\}_{t\geq 0}$ is $\widetilde{\mathbb{P}}$-martingale relative to the filtration $\{\mathcal{F}_W(t)\}_{t\geq 0}$. By Itô's product rule we have

$$\begin{aligned} d(\widetilde{W}(t)Z(t)) &= \widetilde{W}(t)dZ(t) + Z(t)d\widetilde{W}(s) + d\widetilde{W}(t)dZ(t) \\ &= (1 - \theta(t)\widetilde{W}(t))Z(t)dW(t), \end{aligned}$$

that is to say,

$$\widetilde{W}(t)Z(t) = \int_0^t (1 - \widetilde{W}(u)\theta(u))Z(u)dW(u).$$

It follows that the stochastic process $\{Z(t)\widetilde{W}(t)\}_{t\geq 0}$ is a $\mathbb{P}$-martingale relative to $\{\mathcal{F}_W(t)\}_{t\geq 0}$, i.e.,

$$\mathbb{E}[Z(t)\widetilde{W}(t)|\mathcal{F}_W(s)] = Z(s)\widetilde{W}(s).$$

But according to (8),

$$\mathbb{E}[Z(t)\widetilde{W}(t)|\mathcal{F}_W(s)] = Z(s)\widetilde{\mathbb{E}}[\widetilde{W}(t)|\mathcal{F}_W(s)].$$

Hence $\widetilde{\mathbb{E}}[\widetilde{W}(t)|\mathcal{F}_W(s)] = \widetilde{W}(s)$, as claimed. $\square$

Conventions

From now on we assume that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the Brownian motion $\{W(t)\}_{t\geq 0}$ are given. Moreover, in order to avoid the need of repeatedly specifying technical assumptions, we stipulate the following conventions:

- All stochastic processes in this chapter are assumed to belong to the space $\mathcal{C}^0[\mathcal{F}_W(t)]$, i.e., they are adapted to $\{\mathcal{F}_W(t)\}_{t\geq 0}$ and have a.s. continuous paths. This assumption may be relaxed, but for our applications it is general enough.
- All Itô integrals in this chapter are assumed to be martingales, which holds for instance when the integrand stochastic process is in the space $\mathbb{L}^2[\mathcal{F}_W(t)]$.

3 Arbitrage-free markets

Consider the 1+1 dimensional market

$$dS(t) = \mu(t)S(t)\,dt + \sigma(t)S(t)\,dW(t), \quad dB(t) = -B(t)r(t)\,dt.$$

The ultimate purpose of this section is to prove that any self-financing portfolio $\{h_S(t), h_B(t)\}_{t\geq 0}$ invested in this market is not an arbitrage. To this purpose we shall use the following simple result:

Theorem 4. *Let a portfolio be given with value $\{V(t)\}_{t\geq 0}$. If there exists a measure $\widetilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ and a filtration $\{\mathcal{F}(t)\}_{t\geq 0}$ such that the discounted value of the portfolio $\{V^*(t)\}_{t\geq 0}$ is a martingale, then for all $T > 0$ the portfolio is not an arbitrage in the interval $[0, T]$.*

Proof. The assumption is that

$$\widetilde{\mathbb{E}}[D(t)V(t)|\mathcal{F}(s)] = D(s)V(s), \quad \text{for all } 0 \leq s \leq t.$$

Since martingales have constant expectation, then $\widetilde{\mathbb{E}}[D(t)V(t)] = \widetilde{\mathbb{E}}[D(0)V(0)] = \widetilde{\mathbb{E}}[V(0)]$.

Assume that the portfolio is an arbitrage in some interval $[0, T]$. Then $V(0) = 0$ almost surely in both probabilities $\mathbb{P}$ and $\widetilde{p}$; as $V^*(0) = V(0)$, then

$$\widetilde{\mathbb{E}}[V^*(t)] = 0, \quad \text{for all } t \geq 0. \tag{10}$$

Moreover $\mathbb{P}(V(T) \geq 0) = 1$ and $\mathbb{P}(V(T) > 0) > 0$. Since $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent, we also have $\widetilde{\mathbb{P}}(V(T) \geq 0) = 1$ and $\widetilde{\mathbb{P}}(V(T) > 0) > 0$. Since the discount process is positive, we also have $\widetilde{\mathbb{P}}(V^*(T) \geq 0) = 1$ and $\widetilde{\mathbb{P}}(V^*(T) > 0) > 0$. However this contradicts (10). Hence our original hypothesis that the portfolio is an arbitrage portfolio is false. $\square$

In view of the previous theorem, to show that self-financing portfolios invested in a 1+1 dimensional market are not arbitrage portfolios we may show that there exists a probability measure $\widetilde{\mathbb{P}}$, equivalent to $\mathbb{P}$, with respect to which the discounted value of such portfolios is a martingale.

We first define such a measure. We have seen that given a stochastic process $\{\theta(t)\}_{t\geq 0}$ satisfying the Novikov condition, the stochastic process $\{Z(t)\}_{t\geq 0}$ defined by

$$Z(t) = \exp\left(-\int_0^t \theta(s)\, dW(s) - \frac{1}{2}\int_0^t \theta(s)^2\, ds\right) \tag{11}$$

is a $\mathbb{P}$-martingale relative to the filtration $\{\mathcal{F}_W(t)\}_{t\geq 0}$ and that the map $\widetilde{\mathbb{P}} : \mathcal{F} \to [0,1]$ given by

$$\widetilde{\mathbb{P}}(A) = \mathbb{E}[Z(T)\mathbb{I}_A] \tag{12}$$

is a probability measure equivalent to $\mathbb{P}$, for all given $T > 0$.

Definition 1. *Consider the 1+1 dimensional market*

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW(t), \quad dB(t) = B(t)r(t)dt.$$

Assume that $\sigma(t) > 0$ almost surely for all times. Let $\{\theta(t)\}_{t\geq 0}$ be the stochastic process given by

$$\theta(t) = \frac{\mu(t) - r(t)}{\sigma(t)}, \tag{13}$$

and define $\{Z(t)\}_{t\geq 0}$ by (11). Assume that $\{\theta(t)\}_{t\geq 0}$ satisfies the Novikov condition, so that $\{Z(t)\}_{t\geq 0}$ is a martingale.

The probability measure $\widetilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ given by (12) is called the **risk-neutral probability measure** *of the market at time T, while the process $\{\theta(t)\}_{t\geq 0}$ is called the* **market price of risk**.

By the definition (13) of the stochastic process $\{\theta(t)\}_{t\geq 0}$, we can rewrite $dS(t)$ as

$$dS(t) = r(t)S(t)dt + \sigma(t)S(t)d\widetilde{W}(t), \tag{14}$$

where

$$d\widetilde{W}(t) = dW(t) + \theta(t)dt. \tag{15}$$

By Girsanov theorem , the stochastic process $\{\widetilde{W}(t)\}_{t\geq 0}$ is a $\widetilde{\mathbb{P}}$-Brownian motion. Moreover, $\{\mathcal{F}_W(t)\}_{t\geq 0}$ is a non-anticipating filtration for $\{\widetilde{W}(t)\}_{t\geq 0}$.

We also recall that a portfolio $\{h_S(t), h_B(t)\}_{t\geq 0}$ is self-financing if its value $\{V(t)\}_{t\geq 0}$ satisfies

$$dV(t) = h_S(t)dS(t) + h_B(t)dB(t), \tag{16}$$

Moreover $S^*(t) = D(t)S(t)$ is the discounted price (at time $t = 0$) of the stock, where $D(t) = \exp(-\int_0^t r(s)\, ds)$ is the discount process.

Theorem 5. *Consider the 1+1 dimensional market*

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW(t), \quad dB(t) = B(t)r(t)dt, \tag{17}$$

where $\sigma(t) > 0$ almost surely for all times.

(i) The discounted stock price $\{S^*(t)\}_{t\geq 0}$ is a $\widetilde{\mathbb{P}}$-martingale in the filtration $\{\mathcal{F}_W(t)\}_{t\geq 0}$.

(ii) A portfolio process $\{h_S(t), h_B(t)\}_{t\geq 0}$ is self-financing if and only if its discounted value satisfies

$$V^*(t) = V(0) + \int_0^t D(s)h_S(s)\sigma(s)S(s)d\widetilde{W}(s). \tag{18}$$

In particular the discounted value of self-financing portfolios is a $\widetilde{\mathbb{P}}$-martingale in the filtration $\{\mathcal{F}_W(t)\}_{t\geq 0}$.

(iii) If $\{h_S(t), h_B(t)\}_{t\geq 0}$ is a self-financing portfolio, then $\{h_S(t), h_B(t)\}_{t\geq 0}$ is not an arbitrage.

Proof. (i) By (14) and $dD(t) = -D(t)r(t)dt$ we have

$$\begin{aligned}
dS^*(t) &= S(t)dD(t) + D(t)dS(t) + dD(t)dS(t) \\
&= -S(t)r(t)D(t)\,dt + D(t)(r(t)S(t)\,dt + \sigma(t)S(t)\,d\widetilde{W}(t)) \\
&= D(t)\sigma(t)S(t)d\widetilde{W}(t),
\end{aligned}$$

and so the discounted price $\{S^*(t)\}_{t\geq 0}$ of the stock is a $\widetilde{\mathbb{P}}$-martingale relative to $\{\mathcal{F}_W(t)\}_{t\geq 0}$.

(ii) By (14) and $h_S(t)S(t) + h_B(t)B(t) = V(t)$, the definition (16) of self-financing portfolio is equivalent to

$$dV(t) = h_S(t)S(t)[(\mu(t) - r(t))dt + \sigma(t)dW(t)] + V(t)r(t)dt. \tag{19}$$

Hence

$$dV(t) = h_S(t)S(t)\sigma(t)d\widetilde{W}(t) + V(t)r(t)dt.$$

In terms of the discounted portfolio value $V^*(t) = D(t)V(t)$ the previous equation reads

$$\begin{aligned}
dV^*(t) &= V(t)dD(t) + D(t)dV(t) + dD(t)dV(t) \\
&= -D(t)V(t)r(t)\,dt + D(t)h_S(t)S(t)\sigma(t)d\widetilde{W}(t) + D(t)V(t)r(t)dt \\
&= D(t)h_S(t)S(t)\sigma(t)d\widetilde{W}(t),
\end{aligned}$$

which proves (18).

(iii) By (18), the discounted value of self-financing portfolios is a $\widetilde{\mathbb{P}}$-martingale relative to the filtration $\{\mathcal{F}_W(t)\}_{t\geq 0}$. As $\widetilde{\mathbb{P}}$ and $\mathbb{P}$ are equivalent, (iii) follows by Theorem 4. $\square$