Financial derivatives and PDE's
Lecture 11

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The risk-neutral pricing formula

Consider the European derivative with pay-off Y and time of maturity $T > 0$.

We assume that Y is $\mathcal{F}_W(T)$-measurable.

Suppose that the derivative is sold at time $t < T$ for the price $\Pi_Y(t)$.

The first concern of the seller is to hedge the derivative, that is to say, to invest the amount $\Pi_Y(t)$ in such a way that the value of the seller portfolio at time T is enough to pay-off the buyer of the derivative.

The purpose of this section is to define a theoretical price for the derivative which makes it possible for the seller to set-up an hedging portfolio. We argue under the following assumptions:

1. the seller is only allowed to invest the amount $\Pi_Y(t)$ in the 1+1 dimensional market consisting of the underlying stock and the risk-free asset (**Δ-hedging**);
2. the investment strategy of the seller is self-financing.

It follows that the sought hedging portfolio is not an arbitrage.

We may interpret this fact as a "fairness" condition on the price of the derivative $\Pi_Y(t)$. In fact, if the seller can hedge the derivative and still be able to make a risk-less profit on the underlying stock, this may be considered unfair for the buyer.

Consider the 1+1 dimensional market

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW(t), \quad dB(t) = B(t)r(t)dt,$$

where $\sigma(t) > 0$ almost surely for all times.

Let $\{h_S(t), h_B(t)\}_{t\geq 0}$ be a self-financing portfolio invested in this market and let $\{V(t)\}_{t\geq 0}$ be its value.

The discounted value $\{V^*(t)\}_{t\geq 0}$ of the portfolio is a $\widetilde{\mathbb{P}}$-martingale relative to the filtration $\{\mathcal{F}_W(t)\}_{t\geq 0}$, hence

$$D(t)V(t) = \widetilde{\mathbb{E}}[D(T)V(T)|\mathcal{F}_W(t)].$$

Requiring the hedging condition $V(T) = Y$ gives

$$V(t) = \frac{1}{D(t)}\widetilde{\mathbb{E}}[D(T)Y|\mathcal{F}_W(t)].$$

Since $D(t)$ is $\mathcal{F}_W(t)$-measurable, we can move it inside the conditional expectation and write the latter equation as

$$V(t) = \widetilde{\mathbb{E}}[Y\frac{D(T)}{D(t)}|\mathcal{F}_W(t)] = \widetilde{\mathbb{E}}[Y\exp(-\int_t^T r(s)\,ds)|\mathcal{F}_W(t)],$$

where we used the definition $D(t) = \exp(-\int_0^t r(s)\,ds)$ of the discount process.

Assuming that the derivative is sold at time t for the price $\Pi_Y(t)$, then the value of the seller portfolio at this time is precisely equal to the premium $\Pi_Y(t)$, which leads to the following definition.

Definition 1. *Let Y be a $\mathcal{F}_W(T)$-measurable random variable with finite expectation. The* **risk-neutral price** *(or* **fair price**, *or* **arbitrage-free price***) at time $t \in [0,T]$ of the European derivative with pay-off Y and time of maturity $T > 0$ is given by*

$$\Pi_Y(t) = \widetilde{\mathbb{E}}[Y\exp(-\int_t^T r(s)\,ds)|\mathcal{F}_W(t)], \tag{1}$$

i.e., it is equal to the value at time t of any self-financing hedging portfolio invested in the underlying stock and the risk-free asset.

Theorem 1. *Consider the 1+1 dimensional market*

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW(t), \quad dB(t) = B(t)r(t)dt,$$

where $\sigma(t) > 0$ almost surely for all times. Assume that the European derivative on the stock with pay-off Y and time of maturity $T > 0$ is priced by (1) and let $\Pi_Y^*(t) = D(t)\Pi_Y(t)$ be the discounted price of the derivative. Then the following holds.

(i) The process $\{\Pi_Y^*(t)\}_{t\in[0,T]}$ is a $\widetilde{\mathbb{P}}$-martingale relative to $\{\mathcal{F}_W(t)\}_{t\geq 0}$.

(ii) There exists a stochastic process $\{\Delta(t)\}_{t\in[0,T]}$, adapted to $\{\mathcal{F}_W(t)\}_{t\geq 0}$, such that

$$\Pi_Y^*(t) = \Pi_Y(0) + \int_0^t \Delta(s)d\widetilde{W}(s), \quad t \in [0,T]. \tag{2}$$

(iii) The portfolio $\{h_S(t), h_B(t)\}_{t\in[0,T]}$ given by

$$h_S(t) = \frac{\Delta(t)}{D(t)\sigma(t)S(t)}, \quad h_B(t) = (\Pi_Y(t) - h_S(t)S(t))/B(t) \tag{3}$$

is self-financing and replicates the derivative at any time, i.e., its value $V(t)$ is equal to $\Pi_Y(t)$ for all $t \in [0,T]$. In particular, $V(T) = \Pi_Y(T) = Y$, i.e., the portfolio is hedging the derivative.

Proof. (i) We have

$$\Pi_Y^*(t) = D(t)\Pi_Y(t) = \widetilde{\mathbb{E}}[\Pi_Y(T)D(T)|\mathcal{F}_W(t)] = \widetilde{\mathbb{E}}[\Pi_Y^*(T)|\mathcal{F}_W(t)],$$

where we used that $\Pi_Y(T) = Y$. Hence, for $s \leq t$, and using Theorem **??**(v),

$$\widetilde{\mathbb{E}}[\Pi_Y^*(t)|\mathcal{F}_W(s)] = \widetilde{\mathbb{E}}[\widetilde{\mathbb{E}}[\Pi_Y^*(T)|\mathcal{F}_W(t)]|\mathcal{F}_W(s)] = \widetilde{\mathbb{E}}[\Pi_Y^*(T)|\mathcal{F}_W(s)] = \Pi_Y^*(s).$$

This shows that the discounted price of the derivative is a $\widetilde{\mathbb{P}}$-martingale relative to the filtration $\{\mathcal{F}_W(t)\}_{t\geq 0}$.

(ii) By (i) we have

$$Z(s)\Pi_Y^*(s) = Z(s)\widetilde{\mathbb{E}}[\Pi_Y^*(t)|\mathcal{F}_W(s)] = \mathbb{E}[Z(t)\Pi_Y^*(t)|\mathcal{F}_W(s)], \tag{4}$$

i.e., the stochastic process $\{Z(t)\Pi_Y^*(t)\}_{t\in[0,T]}$ is a $\mathbb{P}$-martingale relative to the filtration $\{\mathcal{F}_W(t)\}_{t\geq 0}$.

Hence, by the martingale representation theorem, there exists a stochastic process $\{\Gamma(t)\}_{t\in[0,T]}$ adapted to $\{\mathcal{F}_W(t)\}_{t\geq 0}$ such that

$$Z(t)\Pi_Y^*(t) = \Pi_Y(0) + \int_0^t \Gamma(s)dW(s), \quad t \in [0,T],$$

i.e.,

$$d(Z(t)\Pi_Y^*(t)) = \Gamma(t)dW(t). \tag{5a}$$

On the other hand, by Itô's product rule,

$$\begin{aligned} d\Pi_Y^*(t) =& d(Z(t)\Pi_Y^*(t)/Z(t)) = d(1/Z(t))Z(t)\Pi_Y^*(t) + 1/Z(t)d(Z(t)\Pi_Y^*(t)) \\ &+ d(1/Z(t))d(Z(t)\Pi_Y^*(t)). \end{aligned} \tag{5b}$$

By Itô's formula and $dZ(t) = -\theta(t)Z(t)dW(t)$, we obtain

$$d(1/Z(t)) = -\frac{1}{Z(t)^2}dZ(t) + \frac{1}{Z(t)^3}dZ(t)dZ(t) = \frac{\theta(t)}{Z(t)}d\widetilde{W}(t). \tag{5c}$$

Hence

$$d(1/Z(t))d(Z(t)\Pi_Y^*(t)) = \frac{\theta(t)\Gamma(t)}{Z(t)}dt. \tag{5d}$$

Combining Equations (5) we have

$$d\Pi_Y^*(t) = \Delta(t)d\widetilde{W}(t), \quad \text{where} \;\; \Delta(t) = \theta(t)\Pi_Y^*(t) + \frac{\Gamma(t)}{Z(t)},$$

which proves (2).

(iii) It is clear that the portfolio $\{h_S(t), h_B(t)\}_{t\in[0,T]}$ given by (3) is adapted to $\{\mathcal{F}_W(t)\}_{t\geq 0}$.

By the definition of $h_B(t)$ we have $V(t) = h_S(t)S(t) + h_B(t)B(t) = \Pi_Y(t)$, hence the portfolio replicates the derivative.

Furthermore (2) entails that $V^*(t) = \Pi_Y^*(t)$ satisfies the assumption in Theorem 6.1(ii) (see previous lecture), hence $\{h_S(t), h_B(t)\}_{t\in[0,T]}$ is a self-financing portfolio, and the proof is completed. $\square$

Put-call parity

Being defined as a conditional expectation, the risk-neutral price (1) can be computed explicitly only for simple models on the market parameters.

However the formula (1) can be used to derive a number of general qualitative properties on the fair price of options.

The most important is the put-call parity relation.

Theorem 2. *Let $\Pi_{\text{call}}(t)$ be the fair price at time t of the European call option on the stock with maturity $T > t$ and strike $K > 0$. Let $\Pi_{\text{put}}(t)$ be the price of the European put option with the same strike and maturity. Then the* **put-call parity identity** *holds:*

$$\Pi_{\text{call}}(t) - \Pi_{\text{put}}(t) = S(t) - KB(t,T), \tag{6}$$

where $B(t,T) = \widetilde{\mathbb{E}}[D(T)/D(t)|\mathcal{F}_W(t)]$ is the fair value at time t of the ZCB with face value=1 and maturity T.

Proof. The pay-off of the call/put option is

$$Y_{\text{call}} = (S(T) - K)_+, \quad Y_{\text{put}} = (K - S(T))_+.$$

Using $(x - K)_+ - (K - x)_+ = (x - K)$, for all $x \in \mathbb{R}$, we obtain

$$\begin{aligned}
\Pi_{\text{call}}(t) - \Pi_{\text{put}}(t) &= \widetilde{\mathbb{E}}[D(t)^{-1}D(T)(S(T) - K)_+|\mathcal{F}_W(t)] - \widetilde{\mathbb{E}}[D(t)^{-1}D(T)(K - S(T))_+|\mathcal{F}_W(t)] \\
&= \widetilde{\mathbb{E}}[D(t)^{-1}D(T)(S(T) - K)|\mathcal{F}_W(t)] \\
&= D(t)^{-1}\widetilde{\mathbb{E}}[D(T)S(T)|\mathcal{F}_W(t)] - K\widetilde{\mathbb{E}}[D(t)^{-1}D(T)|\mathcal{F}_W(t)] \\
&= S(t) - KB(t,T),
\end{aligned}$$

where in the last step we use that the discounted stock price process is a martingale in the risk-neutral probability measure. $\square$