Financial derivatives and PDE's
Lecture 12

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Black-Scholes price of standard European derivatives

In the particular case of a standard European derivative, i.e., when $Y = g(S(T))$, for some measurable function g, the risk-neutral price formula becomes

$$\Pi_Y(t) = \widetilde{\mathbb{E}}[g(S(T))\exp(-\int_t^T r(s)\,ds)|\mathcal{F}_W(t)].$$

Moreover

$$S(T) = S(t)\exp\left(\int_t^T (r(s) - \frac{1}{2}\sigma^2(s))ds + \int_t^T \sigma(s)d\widetilde{W}(s)\right),$$

hence the risk-neutral price of standard European derivatives takes the form

$$\Pi_Y(t) = \widetilde{\mathbb{E}}[g(S(t)e^{\int_t^T (r(s)-\frac{1}{2}\sigma^2(s))ds+\int_t^T \sigma(s)d\widetilde{W}(s)})\exp(-\int_t^T r(s)\,ds)|\mathcal{F}_W(t)]. \tag{1}$$

In this section we compute (1) in a **Black-Scholes market**, i.e., in the case when the market parameters $\{\mu(t)\}_{t\geq 0}$, $\{\sigma(t)\}_{t\geq 0}$, $\{r(t)\}_{t\geq 0}$ are deterministic constants.

Letting $\mu(t) = \mu$, $r(t) = r$, $\sigma(t) = \sigma > 0$ we obtain that the stock price satisfies

$$dS(t) = rS(t)\,dt + \sigma S(t)\,d\widetilde{W}(t), \tag{2}$$

where $\widetilde{W}(t) = W(t) + \frac{\mu - r}{\sigma}\,t$ is a Brownian motion in the risk-neutral probability measure.

The market price of risk $\theta = (\mu - r)/\sigma$ is constant. Integrating (2) we obtain that, in the risk-neutral probability, $S(t)$ is given by the geometric Brownian motion

$$S(t) = S(0)e^{(r-\frac{\sigma^2}{2})t+\sigma\widetilde{W}(t)}. \tag{3}$$

By (1), the risk-neutral price of the standard European derivative is

$$\Pi_Y(t) = e^{-r\tau}\widetilde{\mathbb{E}}[g(S(t)e^{(r-\frac{1}{2}\sigma^2)\tau}e^{\sigma(\widetilde{W}(T)-\widetilde{W}(t))})|\mathcal{F}_W(t)],$$

where $\tau = T - t$ is the time left to maturity.

As the increment $\widetilde{W}(T) - \widetilde{W}(t)$ is independent of $\mathcal{F}_W(t)$ and $S(t)$ is $\mathcal{F}_W(t)$-measurable, the conditional expectation above can be computed using the independence lemma, namely

$$\Pi_Y(t) = v_g(t, S(t)), \tag{4a}$$

where the **Black-Scholes price function** $v_g : \overline{\mathcal{D}_T^+} \to \mathbb{R}$ is given by

$$v_g(t,x) = e^{-r\tau}\widetilde{\mathbb{E}}[g(xe^{(r-\frac{1}{2}\sigma^2)\tau}e^{\sigma(\widetilde{W}(T)-\widetilde{W}(t))})] = \frac{e^{-r\tau}}{\sqrt{2\pi}}\int_{\mathbb{R}} g(xe^{(r-\frac{1}{2}\sigma^2)\tau}e^{\sigma\sqrt{\tau}y})e^{-\frac{y^2}{2}}\,dy. \tag{4b}$$

Of course we need some conditions on the function g in order for the integral in the right-hand side of (4b) to converge to a smooth function. For our purposes it suffices to require that $g \in \mathcal{G}$, where

$$\mathcal{G} = \{g : [0,\infty) \to \mathbb{R} : \begin{array}{l} (i)\ g \text{ is almost everywhere twice differentiable} \\ (ii)\ |g(z)| \leq A + B|z| \text{ for some constants } A, B > 0 \\ (iii)\ g', g'' \text{ are uniformly bounded}\} \end{array} \tag{5}$$

Conditions (i)-(iii) are satisfied by the typical pay-off functions used in the applications. It is an easy exercise to show that $g \in \mathcal{G} \Rightarrow v_g \in C^{1,2}(\mathcal{D}_T^+)$.

Definition 1. *Let $g \in \mathcal{G}$. The stochastic process $\{\Pi_Y(t)\}_{t\in[0,T]}$ given by (4), is called the* **Black-Scholes price** *of the standard European derivative with pay-off $Y = g(S(T))$ and time of maturity $T > 0$.*

Next we show that the formula (4) is equivalent to the Markov property of the geometric Brownian motion (3) in the risk-neutral probability measure $\widetilde{\mathbb{P}}$.

To this purpose we rewrite the Black-Scholes price function as $v_g(t, x) = h(T - t, x)$, where, by a change of variable in the integral on the right hand side of (4b),

$$h(\tau, x) = \int_{\mathbb{R}} g(y)q(\tau, x, y)\, dy,$$

where

$$q(\tau, x, y) = \frac{e^{-r\tau}\mathbb{I}_{y>0}}{y\sqrt{2\pi\sigma^2\tau}} \exp\left[-\frac{1}{2\sigma^2\tau}\left(\log\frac{y}{x} - (r - \frac{1}{2}\sigma^2)\tau\right)^2\right].$$

Letting

$$p_*(\tau, x, y) = e^{r\tau}q(\tau, x, y),$$

we obtain that p_*, the risk-neutral pricing formula of standard European derivatives in a market with constant parameters is equivalent to the identity

$$\widetilde{\mathbb{E}}[g(S(T))|\mathcal{F}_{\widetilde{W}}(t)] = \int_{\mathbb{R}} p_*(T - t, S(t), y)g(y)\, dy,$$

and thus, since $0 \leq t \leq T$ are arbitrary, it is equivalent to the Markov property of the geometric Brownian motion (3) in the risk-neutral probability measure $\widetilde{\mathbb{P}}$, with p_* being the transition density of the geometric Brownian motion (3).

Moreover $u(\tau, x) = \int_{\mathbb{R}} p_*(\tau, x, y)g(y)\, dy$ satisfies the Kolmogorov PDE

$$-\partial_\tau u + rx\partial_x u + \frac{1}{2}\sigma^2 x^2 \partial_x^2 u = 0, \quad u(0, x) = h(0, x) = v_g(T, x) = g(x).$$

Hence the function $h(\tau, x) = e^{-r\tau}u(\tau, x)$ satisfies

$$-\partial_\tau h + rx\partial_x h + \frac{1}{2}\sigma^2 x^2 \partial_x^2 h = rh, \quad h(0,x) = g(x).$$

As $v_g(t,x) = h(T-t,x)$, we obtain the following result.

Theorem 1. *Given $g \in \mathcal{G}$, the Black-Scholes price function v_g is the unique strong solution of the* **Black-Scholes** *PDE*

$$\partial_t v_g + rx\partial_x v_g + \frac{1}{2}\sigma^2 x^2 \partial_x^2 v_g = rv_g, \quad (t,x) \in \mathcal{D}_T^+ \tag{6a}$$

with the **terminal condition**

$$v_g(T,x) = g(x). \tag{6b}$$

Next we consider the problem of constructing a replicating (and thus hedging) portfolio of the derivative.

Theorem 2. *Consider a standard European derivative priced according to Definition 1. The portfolio $\{h_S(t), h_B(t)\}$ given by*

$$h_S(t) = \partial_x v_g(t, S(t)), \quad h_B(t) = (\Pi_Y(t) - h_S(t)S(t))/B(t)$$

is a self-financing replicating portfolio for the derivative.

Proof. We have to show that the discounted value of the Black-Scholes price satisfies

$$d\Pi_Y^*(t) = D(t)S(t)\sigma\partial_x v_g(t,S(t))d\widetilde{W}(t).$$

A straightforward calculation, using $\Pi_Y(t) = v_g(t,S(t))$, Itô's formula and Itô's product rule, gives

$$\begin{aligned} d(D(t)\Pi_Y(t)) =& D(t)[\partial_t v_g(t,x) + rx\partial_x v_g(t,x) + \frac{1}{2}\sigma^2 x^2 \partial_x^2 v_g(t,x) - rv_g(t,x)]_{x=S(t)} \\ &+ D(t)\sigma S(t)\partial_x v_g(t,S(t))d\widetilde{W}(t). \end{aligned} \tag{7}$$

Since v_g solves the Black-Scholes PDE (6a), the result follows. □

Black-Scholes price of European vanilla options

In this section we focus the discussion on call/put options, which are also called **vanilla options**. We thereby assume that the pay-off of the derivative is given by

$$Y = (S(T) - K)_+, \text{ i.e., } Y = g(S(T)),\ g(x) = (x - K)_+, \quad \text{for a call option,}$$

$$Y = (K - S(T))_+, \text{ i.e., } Y = g(S(T)),\ g(x) = (K - x)_+, \quad \text{for a put option.}$$

The function v_g given by (4b) will be denoted by C, for a call option, and by P, for a put option.

Theorem 3. *The Black-Scholes price at time t of a European call option with strike price $K > 0$ and maturity $T > 0$ is given by $C(t, S(t))$, where*

$$C(t, x) = x\Phi(d_1) - Ke^{-r\tau}\Phi(d_2), \tag{8a}$$

$$d_2 = \frac{\log\left(\frac{x}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}, \quad d_1 = d_2 + \sigma\sqrt{\tau}, \tag{8b}$$

and where $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\, dy$ is the standard normal distribution. The Black-Scholes price of the corresponding put option is given by $P(t, S(t))$, where

$$P(t, x) = \Phi(-d_2)Ke^{-r\tau} - \Phi(-d_1)x. \tag{8c}$$

Moreover the put-call parity identity holds:

$$C(t, S(t)) - P(t, S(t)) = S(t) - Ke^{-r\tau}. \tag{9}$$

Proof. We derive the Black-Scholes price of call options only, the argument for put options being similar. We substitute $g(z) = (z - K)_+$ into the right hand side of (4b) and obtain

$$C(t, x) = \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(xe^{(r-\frac{1}{2}\sigma^2)\tau} e^{\sigma\sqrt{\tau}y} - K \right)_+ e^{-\frac{y^2}{2}}\, dy.$$

Now we use that $xe^{(r-\frac{1}{2}\sigma^2)\tau}e^{\sigma\sqrt{\tau}y} > K$ if and only if $y > -d_2$. Hence

$$C(t, x) = \frac{e^{-r\tau}}{\sqrt{2\pi}} \left[\int_{-d_2}^{\infty} xe^{(r-\frac{1}{2}\sigma^2)\tau} e^{\sigma\sqrt{\tau}y} e^{-\frac{y^2}{2}} - K \int_{-d_2}^{\infty} e^{-\frac{y^2}{2}}\, dy \right].$$

Using $-\frac{1}{2}y^2 + \sigma\sqrt{\tau}y = -\frac{1}{2}(y - \sigma\sqrt{\tau})^2 + \frac{\sigma^2}{2}\tau$ and changing variable in the integrals we obtain

$$\begin{aligned}
C(t, x) &= \frac{e^{-r\tau}}{\sqrt{2\pi}} \left[xe^{r\tau} \int_{-d_2}^{\infty} e^{-\frac{1}{2}(y-\sigma\sqrt{\tau})^2}\, dy - K \int_{-d_2}^{\infty} e^{-\frac{y^2}{2}}\, dy \right] \\
&= \frac{e^{-r\tau}}{\sqrt{2\pi}} \left[xe^{r\tau} \int_{-\infty}^{d_2+\sigma\sqrt{\tau}} e^{-\frac{1}{2}y^2}\, dy - K \int_{-\infty}^{d_2} e^{-\frac{y^2}{2}}\, dy \right] \\
&= s\Phi(d_1) - Ke^{-r\tau}\Phi(d_2).
\end{aligned}$$

The put-call parity (9) follows by replacing $r(t) = r$ in (**??**), or directly by (8):

$$\begin{aligned} C(t,x) - P(t,x) &= s\Phi(d_1) - Ke^{-r\tau}\Phi(d_2) - \Phi(-d_2)Ke^{-r\tau} + s\Phi(-d_1) \\ &= x(\Phi(d_1) + \Phi(-d_1)) - Ke^{-r\tau}(\Phi(d_2) + \Phi(-d_2)). \end{aligned}$$

As $\Phi(z) + \Phi(-z) = 1$, the claim follows. □

The greeks. Implied volatility and volatility curve

The Black-Scholes price of a call (or put) option derived in Theorem 3 depends on the price of the underlying stock, the time to maturity, the strike price, as well as on the (constant) market parameters r, σ (it does not depend on α).

The partial derivatives of the price function C with respect to these variables are called **greeks**.

We collect the most important ones (for call options) in the following theorem.

Theorem 4. *The price function C of call options satisfies the following:*

$$\Delta := \partial_x C = \Phi(d_1), \tag{10}$$

$$\Gamma := \partial_x^2 C = \frac{\phi(d_1)}{x\sigma\sqrt{\tau}}, \tag{11}$$

$$\rho := \partial_r C = K\tau e^{-r\tau}\Phi(d_2), \tag{12}$$

$$\Theta := \partial_t C = -\frac{x\phi(d_1)\sigma}{2\sqrt{\tau}} - rKe^{-r\tau}\Phi(d_2), \tag{13}$$

$$\nu := \partial_\sigma C = x\phi(d_1)\sqrt{\tau} \quad \textit{(called “vega”).} \tag{14}$$

In particular:

- $\Delta > 0$, i.e., the price of a call is increasing on the price of the underlying stock;
- $\Gamma > 0$, i.e., the price of a call is convex on the price of the underlying stock;
- $\rho > 0$, i.e., the price of the call is increasing on the risk-free interest rate;
- $\Theta < 0$, i.e., the price of the call is decreasing in time;
- $\nu > 0$, i.e., the price of the call is increasing on the volatility of the stock.

Implied volatility

Let us temporarily re-denote the Black-Scholes price of the call as

$$C(t, S(t), K, T, \sigma),$$

which reflects the dependence of the price on the parameters K, T, σ (we disregard the dependence on r).

As shown in Theorem 4,

$$\partial_\sigma C(t, S(t), K, T, \sigma) = \text{vega} = \frac{S(t)}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \sqrt{\tau} > 0.$$

Hence the Black-Scholes price of the option is an increasing function of the volatility. Furthermore, it can be shown that

$$\lim_{\sigma \to 0^+} C(t, S(t), K, T) = (S(t) - Ke^{-r\tau})_+, \quad \lim_{\sigma \to +\infty} C(t, S(t), K, T) = S(t).$$

Therefore the function $C(t, S(t), K, T, \cdot)$ is a one-to-one map from $(0, \infty)$ into the interval $I = ((S(t) - Ke^{-r\tau})_+, S(t))$.

Now suppose that at some given *fixed* time t the real market price of the call is $\widetilde{C}(t) \in I$.

Then there exists a unique value of σ, which depends on the fixed parameters T, K and which we denote by $\sigma_{\text{imp}}(T, K)$, such that

$$C(t, S(t), K, T, \sigma_{\text{imp}}(T, K)) = \widetilde{C}(t).$$

$\sigma_{\text{imp}}(T, K)$ is called the **implied volatility** of the option.

The implied volatility must be computed numerically (for instance using Newton's method), since there is no close formula for it.

The implied volatility of an option (in this example of a call option) is a very important parameter and it is often quoted together with the price of the option.

If the market followed exactly the assumptions in the Black-Scholes theory, then the implied volatility would be a constant, independent of T, K and equal to the volatility of the underlying asset.

In this respect, $\sigma_{\text{imp}}(T, K)$ may be viewed as a quantitative measure of how real markets deviate from ideal Black-Scholes markets.

Volatility curve

As mentioned before, the implied volatility depends on the parameters T, K. Here we are particularly interested in the dependence on the strike price, hence we re-denote the implied volatility as $\sigma_{\text{imp}}(K)$.

If the market behaved exactly as in the Black-Scholes theory, then $\sigma_{\text{imp}}(K) = \sigma$ for all values of K, hence the graph of $K \to \sigma_{\text{imp}}(K)$ would be just a straight horizontal line.

Given that real markets do not satisfy exactly the assumptions in the Black-Scholes theory, what can we say about the graph of the **volatility curve** $K \to \sigma_{\text{imp}}(K)$?

Remarkably, it has been found that there exists recurrent convex shapes for the graph of volatility curves, which are known as **volatility smile** and **volatility skew**.

In the case of a volatility smile, the minimum of the graph is reached at the strike price $K \approx S(t)$, i.e., when the call is at the money. This behavior indicates that the more the call is far from being at the money, the more it will be overpriced.

Volatility smiles have been common in the market since after the crash in October 19$^{\text{th}}$, 1987 (Black Monday), indicating that this event led investors to be more cautious when trading on options that are in or out of the money.