Lecture 7

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Financial derivatives and PDE's
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$dS(t) = \mu(t) S(t)\, dt + \sigma(t) S(t)\, dW(t)$ IS NOT A SDE

$(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\geq 0}, \mathbb{P})$

$\{W(t)\}_{t\geq 0}$ $\mathbb{P}$ - BM

$\{\mathcal{F}(t)\}_{t\geq 0}$ NON-ANTICIPATING

Stochastic differential equations

Definition 1. *Given* $s \geq 0$, $\alpha, \beta \in C^0([s, \infty) \times \mathbb{R})$, *and a deterministic constant* $x \in \mathbb{R}$, *we say that a stochastic process* $\{X(t)\}_{t\geq s}$ *is a* **global (strong) solution** *to the* **stochastic differential equation (SDE)**

$$dX(t) = \alpha(t, X(t))\, dt + \beta(t, X(t))\, dW(t) \tag{1}$$

NOTATION

with initial value $\boxed{X(s) = x}$ *at time* $t = s$, *if* $\{X(t)\}_{t\geq s} \in \mathcal{C}^0[\mathcal{F}(t)]$ *and*

$$X(t) = x + \int_s^t \alpha(\tau, X(\tau))\, d\tau + \int_s^t \beta(\tau, X(\tau))\, dW(\tau), \quad t \geq s. \tag{2}$$

$t \in [s, \infty)$ (EQUIVALENTLY, $t \in [0, T]$, $\forall T > 0$)

FOR $\beta = 0$, THE SDE REDUCES TO THE ORDINARY DIFF. EQ. (ODE)

$$\frac{dX}{dt} = \alpha(t, X(t)), \quad X(s) = x$$

PICARD THEOREM

Remarks

- The initial value of a SDE can be a random variable instead of a deterministic constant, but we shall not need this more general case.
- The integrals in the right hand side of (2) are well-defined, as the integrand functions have continuous paths a.s.

Of course one needs suitable assumptions on the functions α, β to ensure that there is a (unique) process $\{X(t)\}_{t\geq s}$ satisfying (2).

The precise statement is contained in the following global existence and uniqueness theorem for SDE's, which is reminiscent of the analogous result for ordinary differential equations (Picard's theorem).

ANALOG OF PICARD THEOREM FOR SDE

Theorem 1. *Assume that for each $T > s$ there exist constants $C_T, D_T > 0$ such that α, β satisfy*

GROWTH CONDITION $\longrightarrow$
$$|\alpha(t,x)| + |\beta(t,x)| \leq C_T(1+|x|), \tag{3}$$
LIPSHITZ CONDITION $\longrightarrow$ STRONG
$$|\alpha(t,x) - \alpha(t,y)| + |\beta(t,x) - \beta(t,y)| \leq D_T|x-y|, \tag{4}$$

for all $t \in [s,T]$, $x, y \in \mathbb{R}$. Then there exists a unique global solution $\{X(t)\}_{t\geq s}$ of the SDE (1) with initial value $X(s) = x$. Moreover $\{X(t)\}_{t\geq s} \in \mathbb{L}^2[\mathcal{F}(t)]$. $\rightarrow \mathbb{E}[\int_0^T X^2(t)\,dt] < \infty$ $\forall T > 0$

The solution of (1) with initial initial value x at time $t = s$ will be also denoted by $\{X(t; s, x)\}_{t\geq s}$.

NOTATION

It can be shown that, under the assumptions of Theorem 1, the random variable $X(t; s, x)$ depends (a.s.) continuously on the initial conditions (s, x).

$X(s; s, x) = x$

Remarks

- The uniqueness statement in Theorem 1 is to be understood "up to null sets". Precisely, if $\{X_i(t)\}_{t\geq s}$, $i = 1, 2$ are two solutions with the same initial value x, then

$$\mathbb{P}(\sup_{t\in[s,T]} |X_1(t) - X_2(t)| > 0) = 0, \quad \text{for all } T > s.$$

- If the assumptions of Theorem 1 are satisfied only up to a *fixed* time $T > 0$, then the solution of (1) could *explode* at some finite time in the future of T. For example, the stochastic process given by $X(t) = \log(W(t) + e^x)$ solves (1) with $\alpha = -\exp(-2x)/2$ and $\beta = \exp(-x)$, but only up to the time $T_* = \inf\{t : W(t) = -e^x\} > 0$. Note that T_* is a random variable in this example (more precisely, a **stopping time**). In these notes we are only interested in global solutions of SDE's, hence we require (3)-(4) to hold for all $T > 0$.

- The growth condition (3) alone is sufficient to prove the existence of a global solution to (1). The Lipschitz condition (4) is used to ensure uniqueness

- A weak solution of (1) is a stochastic process $\{X(t)\}_{t\geq s}$ that satisfy (2) for *some* Brownian motion $\{W(t)\}_{t\geq 0}$ (not necessarily equal to the given one).

Exercise 5.1

Within many applications in finance, the drift term $\alpha(t, x)$ is linear, an so it can be written in the form

$$\alpha(t, x) = a(b - x), \quad a, b \text{ constant}. \tag{5}$$

A stochastic process $\{X(t)\}_{t \geq 0}$ is called **mean reverting** if there exists a constant c such that $\mathbb{E}[X(t)] \to c$ as $t \to +\infty$. Most financial variables are required to satisfy the mean reversion property. Assume that β satisfies the assumptions in Theorem 1. Prove that the solution $\{X(t; s, x)\}_{t \geq 0}$ of (1) with linear drift (5) satisfies

$$\mathbb{E}[X(t; s, x)] = xe^{-a(t-s)} + b(1 - e^{-a(t-s)}). \tag{6}$$

Hence the process $\{X(t; s, x)\}_{t \geq 0}$ is mean reverting if and only if $a > 0$ and in this case the long time mean is given by $c = b$.

SOLUTION:

$$dX(t) = a(b - X(t))\,dt + \beta(t, X(t))\,dW(t)$$

$$d(e^{+at}X(t)) = d(e^{+at})X(t) + e^{+at}\,dX(t) + d(e^{+at})\,dX(t) \rightarrow \text{THIS IS ZERO}$$

$$= e^{+at}(+a\,dt)X(t) + e^{+at}\big(ab\,dt - aX(t)\,dt + \beta(t, X(t))\,dW(t)\big)$$

$$= ab\,e^{+at}\,dt + e^{at}\beta(t, X(t))\,dW(t)$$

$$\Rightarrow e^{at}X(t) - e^{as}\underbrace{X(s)}_{x} = ab\underbrace{\int_s^t e^{a\tau}\,d\tau}_{\frac{1}{a}(e^{at} - e^{as})} + \int_s^t \beta(\tau, X(\tau))\,dW(\tau)$$

THIS ITÔ INTEGRAL IS A MARTINGALE

$$X(t) = xe^{a(s-t)} + b(1 - e^{a(s-t)}) + e^{-at}\int_s^t \beta(\tau, X(\tau))\,dW(\tau)$$

$$\leq C(1 + |X(\tau)|)$$

$$\mathbb{E}[X(t)] = xe^{a(s-t)} + b(1 - e^{a(s-t)}) + \mathbb{E}[\cdots] \rightarrow 0$$

Linear SDE's

A SDE of the form

$\alpha(t,x(t))$ $\beta(t,x(t))$

$$dX(t) = (a(t) + b(t)X(t))\,dt + (\gamma(t) + \sigma(t)X(t))\,dW(t), \quad X(s) = x, \qquad (7)$$

where a, b, γ, σ are deterministic functions of time, is called a **linear** stochastic differential equation.

We assume that for all $T > 0$ there exists a constant C_T such that

$$\sup_{t\in[s,T]} (|a(t)| + |b(t)| + |\gamma(t)| + |\sigma(t)|) < C_T,$$

and so by Theorem 1 there exists a unique global solution of (7).

For example, the geometric Brownian motion solves the linear SDE

$$dS(t) = \mu S(t)\,dt + \sigma S(t)\,dW(t),$$

where $\mu = \alpha + \sigma^2/2$.

Linear SDE's can be solved explicitly, as shown in the following theorem.

Theorem 2. *The solution $\{X(t)\}_{t\geq s}$ of (7) is given by $X(t) = Y(t)Z(t)$, where*

$$Z(t) = \exp\left(\int_s^t \sigma(\tau)dW(\tau) + \int_s^t (b(\tau) - \frac{\sigma(\tau)^2}{2})d\tau\right),$$

$$Y(t) = x + \int_s^t \frac{a(\tau) - \sigma(\tau)\gamma(\tau)}{Z(\tau)}\,d\tau + \int_s^t \frac{\gamma(\tau)}{Z(\tau)}\,dW(\tau).$$

PROOF: ONE OF THE EXERCISES IN THE 1st ASSIGNMENT

For example, in the special case in which the functions a, b, γ, σ are constant (independent of time), the solution of (7) with initial value $X(0) = x$ at time $t = 0$ is

$$X(t) = e^{\sigma W(t) + (b - \frac{\sigma^2}{2})t}\Big(x + (a - \gamma\sigma)\int_0^t e^{-\sigma W(\tau) - (b - \frac{\sigma^2}{2})\tau} d\tau + \gamma \int_0^t e^{-\sigma W(\tau) - (b - \frac{\sigma^2}{2})\tau} dW(\tau)\Big).$$

$d(t,x) = a + bx \qquad b(t,x) = \gamma$

Exercise 5.3

Consider the linear SDE (7) with constant coefficients a, b, γ and $\sigma = 0$, namely

$$dX(t) = (a + bX(t))\,dt + \gamma dW(t), \quad t \geq s, \quad X(s) = x. \tag{8}$$

Find the solution and show that $X(t; s, x) \in \mathcal{N}(m(t-s, x), \Delta(t-s)^2)$, where

$$m(\tau, x) = xe^{b\tau} + \frac{a}{b}(e^{b\tau} - 1), \quad \Delta(\tau)^2 = \frac{\gamma^2}{2b}(e^{2b\tau} - 1). \tag{9}$$

$\tau = t - s$

SOLUTION:

$$d(e^{-bt}X(t)) = (-be^{-bt}dt)X(t) + e^{-bt}dX(t) + \cancel{(-be^{-bt}dt)\,dX(t)}$$

$$= \cancel{-be^{-bt}X(t)\,dt} + a e^{-bt}dt + \cancel{be^{-bt}X(t)\,dt} + e^{-bt}\gamma\, dW(t)$$

$$e^{-bt}X(t) = e^{-bs}x + a\int_s^t e^{-b\tau}d\tau + \gamma\int_s^t e^{-b\tau}dW(\tau)$$

$$= xe^{-bs} - \frac{a}{b}(e^{-bt} - e^{-bs}) + \gamma\int_s^t e^{-b\tau}dW(\tau)$$

$$X(t) = x e^{b(t-s)} - \frac{a}{b}(1 - e^{b(t-s)}) + \int_s^t \gamma e^{b(t-\tau)}dW(\tau)$$

BY EXERCISE 4.8 (SEE 1st ASSIGNMENT), THIS ITÔ'S INTEGRAL IS NORMALLY DISTRIBUTED $\Rightarrow$ $X(t)$ IS NORMALLY DISTRIBUTED

MOREOVER $\mathbb{E}[X(t)] = xe^{b\tau} - \frac{a}{b}(1 - e^{b\tau})$, $\tau = t - s$

$$VAR[X(t)] = VAR\left[\int_s^t \gamma e^{b(t-\tau)} dW(\tau)\right] = \{ \text{ITÔ ISOMETRY} \}$$

$$= \int_s^t \gamma^2 e^{2b(t-\tau)} d\tau = (\Delta(t-s))^2 \quad \boxed{}$$

Exercise 5.4

Find the solution $\{X(t)\}_{t\geq 0}$ of the linear SDE

$$\longrightarrow \quad dX(t) = tX(t)\,dt + dW(t), \quad t \geq 0$$

with initial value $X(0) = 1$. Find $\text{Cov}(X(s), X(t))$.

DO IT YOURSELF (SOLUTION IS IN THE BOOK)

SOLUTION:

$$d\left(e^{-\frac{t^2}{2}} X(t)\right) = e^{-\frac{t^2}{2}}(-t)\,dt\,X(t) + e^{-t^2/2}\,dX(t)$$

$$+ \cancel{d\left(e^{-t^2/2}\right) dX(t)} \to 0$$

$$= -t e^{-t^2/2} \cancel{X(t)\,dt} + e^{-t^2/2} \cancel{tX(t)\,dt}$$

$$+ e^{-t^2/2}\,dW(t)$$

$$e^{-t^2/2} X(t) = 1 + \int_0^t e^{-\tau^2/2}\,dW(\tau)$$

$$\Rightarrow X(t) = e^{t^2/2} + \int_0^t e^{t^2/2 - \tau^2/2}\,dW(\tau)$$

$$\Rightarrow X(t) \in N\left(e^{t^2/2}, \int_0^t e^{(t^2-\tau^2)}\,d\tau\right)$$

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$$(1) \begin{cases} dX(t) = \alpha(\cancel{t}, X(t))\,dt + \beta(\cancel{t}, X(t))\,dW(\tau) \\ X(s) = x \end{cases}$$

$$E[g(X(t;s,x)) \mid \mathcal{F}(u)] = f_g(t, u, X(t;s,x))$$

Markov property MARKOV PROPERTY

$X(s) = x$

$\mathbb{E}[g(X(t;s,x)) \mid \mathcal{F}(u)] = f_g(t,u,X(t;s,x))$

Markov property

MARKOV PROPERTY

$s \le u < t$

IMP!! It can be shown that, under the assumptions of Theorem 1, the solution $\{X(t;s,x)\}_{t\geq s}$ of (1) is a Markov process.

τ

TIME

u t

IMP!! Moreover when α, β in (1) are time-independent, $\{X(t;s,x)\}_{t\geq s}$ is a homogeneous Markov process.

$f_g(t,u,x) = f_g(t-u,0,x)$

The fact that solutions of SDE's should satisfy the Markov property is quite intuitive, for, as shown in Theorem 1, the solution at time t is uniquely characterized by the initial value at time $s < t$. Consider for example the linear SDE (8).

As shown in Exercise 5.3, the solution satisfies $X(t;s,x) \in \mathcal{N}(m(t-s,x), \Delta(t-s)^2)$, where $m(\tau,x)$ and $\Delta(\tau)$ are given by (9).

The transition density of the Markov process $\{X(t;s,x)\}_{t\geq 0}$ is given by the pdf of the random variable $X(t;s,x)$, namely $p(t,s,x,y) = p_*(t-s,x,y)$, where

$p(t-s,0,x,y)$

$$p_*(\tau,x,y) = e^{\frac{(y-m(\tau,x))^2}{2\Delta(\tau)^2}} \frac{1}{\sqrt{2\pi\Delta(\tau)^2}}. \tag{10}$$

This example rises the question of how one can find the transition density of the solution to a SDE (assuming that such density exists). This problem will be discussed in the next lecture. IMP!!

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$\boxed{\alpha(t,x)}$

DRIFT $= \boxed{\alpha(t, X(t))}$

α DOES ~~NOT~~ DEPEND ON TIME MEANS THAT $\alpha(t,x) = \alpha(x)$. HOWEVER THE DRIFT STILL DEPENDS ON TIME:

DRIFT $= \alpha(X(t))$

$$dX(t) = \alpha(t, X(t))\,dt + \beta(t, X(t))\,dW(t)$$

$$dX_1(t) = \alpha_1(t, X_1(t), X_2(t))\,dt + \beta_{11}(t, X_1(t), X_2(t))\,dW_1(t) + \beta_{12}(t, X_1(t), X_2(t))\,dW_2(t)$$

$$dX_2(t) = \alpha_2(t, X_1(t), X_2(t))\,dt + \beta_{21}(t, X_1(t), X_2(t))\,dW_1(t) + \beta_{22}(t, X_1(t), X_2(t))\,dW_2(t)$$

Systems of SDE's

Occasionally in the course we need to consider systems of several SDE's.

All the results presented in this section extend *mutatis mutandis* to systems of SDE's, the difference being merely notational.

For example, given two Brownian motions $\{W_1(t)\}_{t\geq 0}$, $\{W_2(t)\}_{t\geq 0}$ and continuous functions $\alpha_1, \alpha_2, \beta_{11}, \beta_{12}, \beta_{21}, \beta_{22} : [s, \infty) \times \mathbb{R}^2 \to \mathbb{R}$, the relations

↳ TYPICALLY ASSUMED TO BE INDEPENDENT

$$\begin{cases} dX_i(t) = \alpha_i(t, X_1(t), X_2(t))\,dt + \sum_{j=1,2} \beta_{ij}(t, X_1(t), X_2(t))dW_j(t), & (11a) \\ X_i(s) = x_i, \quad i = 1, 2 & (11b) \end{cases}$$

$$X_1(s) = x_1, \quad X_2(s) = x_2$$

define a system of two SDE's on the stochastic processes $\{X_1(t)\}_{t\geq 0}$, $\{X_2(t)\}_{t\geq 0}$ with initial values $X_1(s) = x_1$, $X_2(s) = x_2$ at time s. As usual, the correct way to interpret the relations above is in the integral form:

$$X_i(t) = x_i + \int_s^t \alpha_i(\tau, X_1(\tau), X_2(\tau))\,d\tau + \sum_{j=1,2} \int_s^t \beta_{ij}(\tau, X_1(\tau), X_2(\tau))dW_j(\tau) \quad i = 1, 2.$$

Upon defining the vector and matrix valued functions $\alpha = (\alpha_1, \alpha_2)^T$, $\beta = (\beta_{ij})_{i,j=1,2}$, and letting $X(t) = (X_1(t), X_2(t))$, $x = (x_1, x_2)$, $W(t) = (W_1(t), W_2(t))$, we can rewrite (11) as

$$dX(t) = \alpha(t, X(t))\,dt + \beta(t, X(t)) \cdot dW(t), \quad X(s) = x, \tag{12}$$

where $\cdot$ denotes the row by column matrix product.

In fact, every system of any arbitrary number of SDE's can be written in the form (12).

Theorem 1 continues to be valid for systems of SDE's, the only difference being that $|\alpha|$, $|\beta|$ in (3)-(4) stand now for the vector norm of α and for the matrix norm of β.