Financial derivatives and PDE's
Lecture 13

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The Asian option

The Asian call option with **arithmetic average**, strike $K > 0$ and maturity $T > 0$ is the non-standard European style derivative with pay-off

$$Y_{\mathrm{AC}} = \left(\frac{1}{T}\int_0^T S(t)\,dt - K\right)_+ ,$$

while for the Asian put the pay-off is

$$Y_{\mathrm{AP}} = \left(K - \frac{1}{T}\int_0^T S(t)\,dt\right)_+ .$$

We study the Asian option in a Black-Scholes market, i.e., assuming that the market parameters are deterministic constants.

The risk-neutral price at time $t \leq T$ of the Asian call is therefore given by

$$\Pi_{\mathrm{AC}}(t) = e^{-r(T-t)}\widetilde{\mathbb{E}}[(Q(T)/T - K)_+ | \mathcal{F}_W(t)],$$

where

$$Q(t) = \int_0^t S(\tau)\,d\tau, \quad S(\tau) = S(0)e^{(r-\frac{\sigma^2}{2})\tau + \sigma\widetilde{W}(\tau)}.$$

Exercise 1. *Prove the following put-call parity identity for Asian options:*

$$\Pi_{\mathrm{AC}}(t) - \Pi_{\mathrm{AP}}(t) = \frac{Q(t)}{T} e^{-r(T-t)} + \frac{S(t)}{rT}(1 - e^{-r(T-t)}) - K e^{-r(T-t)}. \tag{1}$$

Exercise 2. *Let $r \geq 0$. Prove the following inequalities between the Black -Scholes price of the Asian call and the European call options:*

$$\Pi_{\mathrm{AC}}(0) < \frac{1 - e^{-rT}}{rT} C(0, S_0, K, T).$$

Conclude from this that for $r \geq 0$ the Asian call is less valuable than the European call. HINT: You need the Jensen inequality for integrals: $f(\frac{1}{b-a}\int_a^b g(x)\,dx) \leq \frac{1}{b-a}\int_a^b f(g(x))\,dx$, for all $b > a$ and for all functions f, g such that f is convex.

A simple closed formula for the price of the Asian option with arithmetic average is not available.

In this lecture we discuss two numerical methods to price the Asian option with arithmetic average, namely the finite difference method applied to the pricing PDE and the Monte Carlo method applied to the risk-neutral pricing formula (at time $t = 0$).

The PDE method

To price the Asian option with arithmetic average using the PDE method we first observe that the stochastic processes $\{Q(t)\}_{t\in[0,T]}$, $\{S(t)\}_{t\in[0,T]}$ satisfy the system of SDE's

$$dQ(t) = S(t)\,dt, \quad dS(t) = rS(t)\,dt + \sigma S(t)\,d\widetilde{W}(t).$$

By the Markov property of SDE's it follows that the risk-neutral price of the Asian call satisfies

$$\Pi_{\mathrm{AC}}(t) = c(t, S(t), Q(t)). \tag{2}$$

for some measurable function c.

Theorem 1. *Let $c : [0,T] \times (0,\infty)^2 \to (0,\infty)$ be the strong solution to the terminal value problem*

$$\partial_t c + rx\partial_x c + x\partial_y c + \frac{\sigma^2}{2}x^2\partial_x^2 c = rc, \quad t \in (0,T),\ x, y > 0 \tag{3a}$$
$$c(T, x, y) = (y/T - K)_+, \quad x, y > 0. \tag{3b}$$

Then (2) holds. Moreover the number of shares of the stock in the self-financing hedging portfolio is given by $h_S(t) = \partial_x c(t, S(t), Q(t))$.

Proof. We have

$$d(e^{-rt}c(t, S(t), Q(t)) = e^{-rt}[\partial_t c + rx\partial_x c + x\partial_y c + \frac{\sigma^2}{2}x^2\partial_x^2 c - rc](t, S(t), Q(t))\,dt$$
$$+ e^{-rt}\partial_x c(t, S(t))\sigma S(t) d\widetilde{W}(t).$$

As c satisfies (3), then

$$d(e^{-rt}c(t, S(t), Q(t)) = e^{-rt}\partial_x c(t, S(t), Q(t))\sigma S(t) d\widetilde{W}(t). \tag{4}$$

It follows that the process $\{e^{-rt}c(t,S(t),Q(t))\}_{t\in[0,T]}$ is a $\widetilde{\mathbb{P}}$-martingale relative to $\{\mathcal{F}_W(t)\}_{t\in[0,T]}$. In particular

$$e^{-rT}\widetilde{\mathbb{E}}[c(T,S(T),Q(T))|\mathcal{F}_W(t)] = e^{-rt}c(t,S(t),Q(t)), \quad T\geq t.$$

Using $c(T,S(T),Q(T)) = (Q(T)/T-K)_+$ proves (2). Moreover by (4) the discounted value of the Asian call satisfies

$$\Pi^*_{\mathrm{AC}}(t) = \Pi_{\mathrm{AC}}(0) + \int_0^t e^{-r\tau}\partial_x c(\tau,S(\tau),Q(\tau))\sigma S(t)\, d\widetilde{W}(\tau),$$

hence the number of shares of the stock in the hedging portfolio is $h_S(t) = \partial_x c(t,S(t),Q(t))$.
$\square$

Exercise 3. *Use Theorem 1 to give an alternative proof of the put-call parity* (1).

A simple closed formula solution for the problem (3) is not available, hence one needs to rely on numerical methods to find approximate solutions.

One such method is the finite difference method described in Chapter **??**. To apply this method one needs to specify the boundary conditions for (3) at infinity and for $\{x = 0, y > 0\}$, $\{y = 0, x > 0\}$.

Concerning the boundary condition at $x = 0$, let $\bar{c}(t, y) = c(t, 0, y)$.

Letting $x = 0$ in (3) we obtain that $\bar{c}$ satisfies $\partial_t \bar{c} = r\bar{c}$ and $\bar{c}(T, y) = (y/T - K)_+$, from which we derive the boundary condition

$$c(t, 0, y) = e^{-r(T-t)} \left(\frac{y}{T} - K \right)_+ . \tag{5}$$

As to the boundary condition when $y \to \infty$, we first observe that the Asian put becomes clearly worthless if $Q(t)$ reaches arbitrarily large values.

Hence the put-call parity (1) leads us to impose

$$c(t, x, y) \sim \frac{y}{T} e^{-r(T-t)}, \quad \text{as } y \to \infty, \text{ for all } x > 0 \tag{6}$$

The boundary conditions at $y = 0$ and $x \to \infty$ are not so obvious. The next theorem shows that one can avoid giving these boundary conditions by a suitable variable transformation.

Theorem 2. *Let* $u : [0, \infty) \times \mathbb{R} \to [0, \infty)$ *be the strong solution to the problem*

$$\partial_t u + \frac{\sigma^2}{2} (\gamma(t) - z)^2 \partial_z^2 u = 0, \quad t \in (0, T), z \in \mathbb{R} \tag{7a}$$

$$u(T, z) = (z)_+, \quad \lim_{z \to -\infty} u(t, z) = 0, \ \lim_{z \to \infty} (u(t, z) - z) = 0, \quad t \in [0, T), \tag{7b}$$

where $\gamma(t) = \frac{1 - e^{-r(T-t)}}{rT}$. *Then the function*

$$c(t, x, y) = xu \left(t, \frac{1}{rT}(1 - e^{-r(T-t)}) + \frac{e^{-r(T-t)}}{x} \left(\frac{y}{T} - K\right) \right) \tag{8}$$

solves (3) *as well as* (5)-(6)

The Monte Carlo method

A very popular method to compute numerically the price of non-standard derivatives is the Monte Carlo method. In this section we describe briefly how the method works for Asian options with arithmetic average and leave the generalization to other derivatives as an exercise.

The crude Monte Carlo method

The Monte Carlo method is, in its simplest form, a numerical method to compute the expectation of a random variable. Its mathematical validation is based on the **Law of Large Numbers**, which states the following: Suppose $\{Y_i\}_{i\geq 1}$ is a sequence of i.i.d. random variables with expectation $\mathbb{E}[Y_i] = \mu$. Then the sample average of the first n components of the sequence, i.e.,

$$\overline{Y} = \frac{1}{n}(Y_1 + Y_2 + \cdots + Y_n),$$

converges (in probability) to μ as $n \to \infty$.

The law of large numbers can be used to justify the fact that if we are given a large number of independent trials $Y_1, \ldots, Y_n$ of the random variable Y, then

$$\mathbb{E}[Y] \approx \frac{1}{n}(Y_1 + Y_2 + \cdots + Y_n).$$

To measure how reliable is the approximation of $\mathbb{E}[Y]$ given by the sample average, consider the standard deviation of the trials $Y_1, \ldots, Y_n$:

$$s_Y = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(\overline{Y} - Y_i)^2}.$$

A simple application of the Central Limit Theorem proves that the random variable

$$\frac{\mu - \overline{Y}}{s_Y/\sqrt{n}}$$

converges in distribution to a standard normal random variable. We use this result to show that the true value μ of $\mathbb{E}[Y]$ has about 95% probability to be in the interval

$$[\overline{Y} - 1.96\frac{s_Y}{\sqrt{n}}, \overline{Y} + 1.96\frac{s_Y}{\sqrt{n}}]. \tag{9}$$

Indeed, for n large,

$$\mathbb{P}\left(-1.96 \leq \frac{\mu - \overline{Y}}{s_Y/\sqrt{n}} \leq 1.96\right) \approx \int_{-1.96}^{1.96} e^{-x^2/2}\,\frac{dx}{\sqrt{2\pi}} \approx 0.95.$$

In the applications to options pricing, the random variable Y is the pay-off of a European derivative. Using the Monte Carlo method and the risk-neutral pricing formula, we can approximate the Black-Scholes price at time $t = 0$ of the European derivative with pay-off Y and maturity $T > 0$ with the sample average

$$\Pi_Y(0) = e^{-rT} \frac{Y_1 + \dots Y_n}{n}, \tag{10}$$

where $Y_1, \dots, Y_n$ is a large number of independent trials of the pay-off. Each trial Y_i is determined by a path of the stock price. Letting $0 = t_0 < t_1 < \dots < t_N = T$ be a partition of the interval $[0, T]$ with size $t_i - t_{i-1} = h$, we may construct a sample of n paths of the geometric Brownian motion on the given partition with the following simple Matlab function:

```
function Path=StockPath(s,sigma,r,T,N,n)
h=T/N;
W=randn(n,N);
q=ones(n,N);
Path=s*exp((r-sigma^2/2)*h.*cumsum(q')+sigma*sqrt(h)*cumsum(W'));
Path=[s*ones(1,n);Path];
```

Note carefully that the stock price is modeled as a geometric Brownian motion with mean of log return $\alpha = r - \sigma^2/2$, which means that the geometric Brownian motion is risk-neutral. This is of course correct, since the expectation in (10) that we want to compute is in the risk-neutral probability measure. In the case of the Asian call option with arithmetic average, strike K and maturity T the pay-off is given by

$$Y = \left(\frac{1}{T} \int_0^T S(t)\, dt - K \right)_+ \approx \left(\frac{1}{N} \sum_{i=1}^N S(t_i) - K \right)_+ .$$

The following function computes the approximate price of the Asian option using the Monte Carlo method:

```
function [price, conf95]=MonteCarlo_AC(s,sigma,r,K,T,N,n)
tic
stockPath=StockPath(s,sigma,r,T,N,n);
payOff=max(0,mean(stockPath)-K);
price=exp(-r*T)*mean(payOff);
conf95=1.96*std(payOff)/sqrt(n);
toc
```

The function also return the 95% confidence interval of the result. For example, by running the command

```
[price, conf95]=MonteCarlo_AC(100,0.5,0.05,100,1/2,100,1000000)
```

we get `price`=8.5799, `conf95`=0.0283, which means that the Black-Scholes price of the Asian option with the given parameters has 95% probability to be in the interval 8.5799 ± 0.0283. The calculation took about 4 seconds. Note that the 95% confidence is $0.0565/8.5799 * 100 \approx 0.66\%$ of the price.

Control variate Monte Carlo method

The crude Monte Carlo method just described can be improved in a number of ways. For instance, it follows by (9) that in order to shrink the confidence interval of the Monte Carlo price one can try to reduce the standard derivation s. There exist several methods to decrease the standard deviation of a Monte Carlo computation, which are collectively called **variance reduction techniques**. Here we describe the **control variate** method.

Suppose we want to compute $\mathbb{E}[Y]$. The idea of the control variate method is to introduce a second random variable Z for which $\mathbb{E}[Z]$ can be computed *exactly* and then write

$$\mathbb{E}[Y] = \mathbb{E}[X] + \mathbb{E}[Z], \quad \text{where } X = Y - Z.$$

Hence the Monte Carlo approximation of $\mathbb{E}[Y]$ can now be written as

$$\mathbb{E}[Y] \approx \frac{X_1 + \cdots + X_n}{n} + \mathbb{E}[Z],$$

where $X_1, \dots, X_n$ are independent trials of the random variable X. This approximation improves the crude Monte Carlo estimate (without control variate) if the sample average estimator of $\mathbb{E}[X]$ is better than the sample average estimator of $\mathbb{E}[Y]$. Because of (9), this will be the case if $(s_X)^2 < (s_Y)^2$. It will now be shown that the latter inequality holds if Y, Z *have a positive large correlation*. Letting $Y_1, \dots, Y_n$ be independent trials of Y and $Z_1, \dots, Z_n$ be independent trials of Z, we compute

$$\begin{aligned}(s_X)^2 &= \frac{1}{n-1} \sum_{i=1}^{n} (\overline{X} - X_i)^2 = \frac{1}{n-1} \sum_{i=1}^{n} ((\overline{Y} - \overline{Z}) - (Y_i - Z_i))^2 \\ &= (s_Y)^2 + (s_Z)^2 - 2C(Y, Z),\end{aligned}$$

where $C(Y, Z)$ is the sample covariance of the trials $(Y_1, \dots, Y_n)$, $(Z_1, \dots, Z_n)$, namely

$$C(Y, Z) = \sum_{i=1}^{n} (\overline{Y} - Y_i)(\overline{Z} - Z_i).$$

Hence $(s_X)^2 < (s_Y)^2$ holds provided $C(Y, Z)$ is sufficiently large and positive (precisely, $C(Y, Z) > s_Z/\sqrt{2}$). As $C(Y, Z)$ is an unbiased estimator of $\mathrm{Cov}(Y, Z)$, then the use of the control variate Z will improve the performance of the crude Monte Carlo method if Y, Z have a positive large correlation.