Financial derivatives and PDE's
Lecture 14

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Local and Stochastic volatility models

In this lecture we present a method to compute the risk-neutral price of European derivatives when the market parameters are not deterministic functions.

We first assume that the interest rate of the money market is constant, i.e., $r(t) = r$, which is quite reasonable for derivatives with short maturity such as options; stochastic interest rate models are important for pricing derivatives with long time of maturity, e.g. coupon bonds.

Assuming that the derivative is the standard European derivative with pay-off function g and maturity T, the risk-neutral price formula becomes

$$\Pi_Y(t) = e^{-r\tau}\widetilde{\mathbb{E}}[g(S(T))|\mathcal{F}_W(t)], \quad \tau = T - t. \tag{1}$$

Motivated by our earlier results on the Black-Scholes price and we attempt to re-write the risk-neutral price formula in the form

$$\Pi_Y(t) = v_g(t, S(t)) \quad \text{for all } t \in [0, T], \text{ for all } T > 0, \tag{2}$$

for some function $v_g : \overline{\mathcal{D}_T^+} \to (0, \infty)$, which we call the pricing function of the derivative.

By (1), this is equivalent to

$$\widetilde{\mathbb{E}}[g(S(T))|\mathcal{F}_W(t)] = e^{r\tau} v_g(t, S(t)) \tag{3}$$

i.e., to the property that $\{S(t)\}_{t\geq 0}$ is a Markov process in the risk-neutral probability measure $\widetilde{\mathbb{P}}$, relative to the filtration $\{\mathcal{F}_W(t)\}_{t\geq 0}$.

At this point it remains to understand for which stochastic processes $\{\sigma(t)\}_{t\geq 0}$ does the generalized geometric Brownian motion satisfies this Markov property.

We have seen that this holds in particular when $\{S(t)\}_{t\geq 0}$ satisfies a (system of) stochastic differential equation(s).

Next we discuss two examples which encompass most of the volatility models used in the applications: Local volatility models and Stochastic volatility models.

Local volatility models

A **local volatility model** is a special case of the generalized geometric Brownian motion in which the instantaneous volatility of the stock $\{\sigma(t)\}_{t\geq 0}$ is assumed to be a deterministic function of the stock price $S(t)$.

Given a continuous function $\beta : [0,\infty) \times [0,\infty) \to (0,\infty)$, we then let

$$\sigma(t)S(t) = \beta(t, S(t)), \tag{4}$$

(and $r(t) = r$) into the geometric Brownian motion, so that the stock price process $\{S(t)\}_{t\geq 0}$ satisfies the SDE

$$dS(t) = rS(t)\,dt + \beta(t, S(t))d\widetilde{W}(t), \quad S(0) = S_0 > 0. \tag{5}$$

We assume that this SDE admits a unique global solution.

In the following we shall also assume that the solution $\{S(t)\}_{t\geq 0}$ of (5) is non-negative a.s. for all $t > 0$.

Note however that the stochastic process solution of (5) will in general hit zero with positive probability at any finite time.

For example, letting $\beta(t,x) = \sqrt{x}$, the stock price (5) is a CIR process with $b = 0$ and so, according to Theorem **??**, $S(t) = 0$ with positive probability for all $t > 0$.

Theorem 1. *Let $g \in \mathcal{G}$ and assume that the Kolmogorov PDE*

$$\partial_t u + rx\partial_x u + \frac{1}{2}\beta(t,x)^2\partial_x^2 u = 0 \quad (t,x) \in \mathcal{D}_T^+, \tag{6}$$

associated to (5) *admits a (necessarily unique) strong solution in the region* $\mathcal{D}_T^+$ *satisfying* $u(T,x) = g(x)$*. Let also*

$$v_g(t,x) = e^{-r\tau}u(t,x).$$

Then we have the following.

(i) v_g *satisfies*

$$\partial_t v_g + rx\partial_x v_g + \frac{1}{2}\beta(t,x)^2\partial_x^2 v_g = rv_g \quad (t,x) \in \mathcal{D}_T^+, \tag{7}$$

and the terminal condition

$$v_g(T,x) = g(x). \tag{8}$$

(ii) The price of the European derivative with pay-off $Y = g(S(T))$ *and maturity* $T > 0$ *is given by* (2)*.*

(iii) The portfolio given by

$$h_S(t) = \partial_x v_g(t, S(t)), \quad h_B(t) = (\Pi_Y(t) - h_S(t)S(t))/B(t)$$

is a self-financing hedging portfolio.

Proof. (i) It is straightforward to verify that v_g satisfies (7).

(ii) Let $X(t) = v_g(t, S(t))$. By Itô's formula we find

$$\begin{aligned} dX(t) &= (\partial_t v_g(t,S(t)) + rS(t)\partial_x v_g(t,S(t)) + \frac{1}{2}\beta(t,S(t))^2\partial_x^2 v_g(t,S(t)))dt \\ &\quad + \beta(t,S(t))\partial_x v_g(t,S(t))d\widetilde{W}(t). \end{aligned}$$

Hence

$$\begin{aligned} d(e^{-rt}X(t)) &= e^{-rt}(\partial_t v_g + rx\partial_x v_g + \frac{1}{2}\beta(t,x)^2\partial_x^2 v_g - rv_g)(t,S(t))dt \\ &\quad + e^{-rt}\beta(t,S(t))\partial_x v_g(t,S(t))d\widetilde{W}(t). \end{aligned}$$

As $v_g(t,x)$ satisfies (7), the drift term in the right hand side of the previous equation is zero. Hence

$$e^{-rt}v_g(t,S(t)) = v_g(t,S_0) + \int_0^t e^{-ru}\beta(u,S(u))\partial_x v_g(u,S(u))d\widetilde{W}(u). \tag{9}$$

It follows that[1] the stochastic process $\{e^{-rt}v_g(t,S(t))\}_{t\geq 0}$ is a $\widetilde{\mathbb{P}}$-martingale relative to $\{\mathcal{F}_W(t)\}_{t\geq 0}$. Hence

$$\widetilde{\mathbb{E}}[e^{-rT}v_g(T,S(T))|\mathcal{F}_W(t)] = e^{-rt}v_g(t,S(t)), \quad \text{for all } 0 \leq t \leq T.$$

[1]Recall that we assume that Itô's integrals are martingales!

Using the boundary condition (8), we find

$$v_g(t, S(t)) = e^{-r\tau}\widetilde{\mathbb{E}}[g(S(T))|\mathcal{F}_W(t)],$$

which proves (2).

(iii) Replacing $\Pi_Y(t) = v_g(t, S(t))$ into (9), we find

$$\Pi_Y^*(t) = \Pi_Y(0) + \int_0^t e^{-ru}\beta(u, S(u))\partial_x v_g(u, S(u))d\widetilde{W}(u).$$

Hence the claim on the hedging portfolio follows.

□

Example: The CEV model

For the **constant elasticity variance (CEV)** model, we have $\beta(t, S(t)) = \sigma S(t)^\delta$, where $\sigma > 0$, $\delta > 0$ are constants.

The SDE for the stock price becomes

$$dS(t) = rS(t)dt + \sigma S(t)^\delta d\widetilde{W}(t), \quad S(0) = S_0 > 0. \tag{10}$$

For $\delta = 1$ we recover the Black-Scholes model.

For $\delta \neq 1$, we can construct the solution of (10) using a CIR process, as shown in the following exercise.

Exercise 1. *Given σ, r and $\delta \neq 1$, define*

$$a = 2r(\delta - 1),\ c = -2\sigma(\delta - 1),\ b = \frac{\sigma^2}{2r}(2\delta - 1),\ \theta = -\frac{1}{2(\delta - 1)}.$$

Let $\{X(t)\}_{t\geq 0}$ be the CIR process

$$dX(t) = a(b - X(t))\,dt + c\sqrt{X(t)}d\widetilde{W}(t), \quad X(0) = x > 0.$$

Show that $S(t) = X(t)^\theta$ solves (10) with $S_0 = x^\theta$.

It follows by Exercise 1, and by Feller's condition $ab \geq c^2/2$ for the positivity of the CIR process, that the solution of (10) remains strictly positive a.s. if $\delta \geq 1$, while for $0 < \delta < 1$, the stock price hits zero in finite time with positive probability.

The Kolmogorov PDE (6) associated to the CEV model is

$$\partial_t u + rx\partial_x u + \frac{\sigma^2}{2} x^{2\delta} \partial_x^2 u = 0, \ \ (t, x) \in \mathcal{D}_T^+.$$

Given a terminal value g at time T as in Theorem 1, the previous equation admits a unique solution.

However a fundamental solution, in the sense of exists only for $\delta > 1$, as otherwise the stochastic process $\{S(t)\}_{t\geq 0}$ hits zero at any finite time with positive probability and therefore the density of the random variable $S(t)$ has a discrete part.