

Financial derivatives and PDE's

Lecture 17

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February 15th, 2021

The lookback option

Lookback options are non-standard European style derivatives whose pay-off depends on the minimum or maximum of the stock price within a given time period until maturity. There exists four main types of lookback options.

- A lookback **call** option with **floating strike** and maturity $T > 0$ gives to the owner the right to buy the underlying stock at maturity for the minimum price of the stock in the interval $[0, T]$. Thus the pay-off for this lookback option is

$$Y_{LC}^{\text{float}} = S(T) - \min\{S(t), t \in [0, T]\}.$$

- A lookback **put** option with **floating strike** and maturity $T > 0$ gives to the owner the right to sell the underlying stock at maturity for the maximum price of the stock in the interval $[0, T]$. Thus the pay-off for this lookback option is

$$Y_{LP}^{\text{float}} = \max\{S(t), t \in [0, T]\} - S(T).$$

- A lookback **call** option with **fixed strike** $K > 0$ and maturity $T > 0$ pays the buyer the difference between the maximum of the stock price in the interval $[0, T]$ and the strike K , provided this difference is positive. Hence the pay-off for this lookback option is

$$Y_{LC}^{\text{fixed}} = (\max\{S(t), t \in [0, T]\} - K)_+.$$

- A lookback **put** option with **fixed strike** $K > 0$ and maturity $T > 0$ pays the buyer the difference between the strike price K and the minimum of the stock price in the interval $[0, T]$, provided this difference is positive. Hence the pay-off for this lookback option is

$$Y_{LP}^{\text{fixed}} = (K - \min\{S(t), t \in [0, T]\})_+.$$

In the following section we focus on the lookback call option with floating strike in a Black-Scholes market. In particular the stock price is given by

$$S(t) = S(0)e^{(r - \frac{1}{2}\sigma^2)t + \sigma\widetilde{W}(t)}$$

and the risk-neutral price for the floating strike lookback call option with maturity $T > 0$ is

$$\Pi_{\text{LC}}^{\text{float}}(t) = e^{-r(T-t)}\widetilde{\mathbb{E}}[S(T) - \min\{S(\tau), \tau \in [0, T]\} | \mathcal{F}_W(t)].$$

Exercise 1. *Show that the price at time $t = 0$ of the lookback call and put options with floating strike is a linear increasing function of the stock price at time zero.*

Pricing PDE for the lookback call option with floating strike

The main purpose of this section is to derive the PDE satisfied by the pricing function of lookback call options with floating strike.

Theorem 1. *Let $v : (0, T) \times (0, \infty) \times (0, \infty)$, $v = v(t, x, y)$, satisfy*

$$\partial_t v + rx\partial_x v + \frac{1}{2}\sigma^2 x^2 \partial_x^2 v = rv, \quad t \in (0, T), \quad x > 0, \quad 0 < y < x, \quad (1a)$$

$$\partial_y v(t, x, x) = 0, \quad t \in [0, T], \quad x > 0, \quad (1b)$$

$$v(T, x, y) = x - y, \quad 0 \leq y \leq x. \quad (1c)$$

Then $\Pi_{\text{LC}}^{\text{float}}(t) = v(t, S(t), \min_{0 \leq \tau \leq t} S(\tau))$.

Proof. Let $Y(t) = \min_{0 \leq \tau \leq t} S(\tau)$; note that $\{Y(t)\}_{t \geq 0}$ is a non-increasing process in the space $\mathcal{C}^0[\mathcal{F}_W(t)]$. However $\{Y(t)\}_{t \geq 0}$ is *not* a diffusion process.

We now show that

$$dY(t)dY(t) = 0.$$

Recall that this means that the quadratic variation of $\{Y(t)\}_{t \geq 0}$ is zero in any interval $[0, T]$ along any sequence of partitions $\{\Pi_n\}_{n \in \mathbb{N}}$ of this interval such that $\|\Pi_n\| \rightarrow 0$, as $n \rightarrow \infty$.

Letting $\Pi_n = \{t_0 = 0, t_1^{(n)}, \dots, t_{m(n)}^n = T\}$, we have to prove that

$$\lim_{n \rightarrow \infty} \sum_{j=1}^{m(n)} (Y(t_j^{(n)}) - Y(t_{j-1}^{(n)}))^2 = 0 \quad \text{in } L^2(\Omega).$$

But

$$\begin{aligned} \sum_{j=1}^{m(n)} (Y(t_j^{(n)}) - Y(t_{j-1}^{(n)}))^2 &\leq \max_j |Y(t_j^{(n)}) - Y(t_{j-1}^{(n)})| \sum_j |Y(t_j^{(n)}) - Y(t_{j-1}^{(n)})| \\ &= \max_j |Y(t_{j-1}^{(n)}) - Y(t_j^{(n)})| \sum_j (Y(t_{j-1}^{(n)}) - Y(t_j^{(n)})) \\ &= \max_j |Y(t_{j-1}^{(n)}) - Y(t_j^{(n)})| (Y(0) - Y(T)), \end{aligned}$$

where in the sum we used that $Y(t)$ is non-increasing to write $|Y(t) - Y(s)| = Y(s) - Y(t)$, for $t \geq s$.

As Y is continuous in time, then $\max_j |Y(t_{j-1}^{(n)}) - Y(t_j^{(n)})| \rightarrow 0$, pointwise in $\omega \in \Omega$.

As $Y(t_j^{(n)}) \leq Y(0)$, then, by the dominated convergence theorem, the limit is zero also in L^2 , which completes the proof of $dY(t)dY(t) = 0$.

Similarly one can prove that $dS(t)dY(t) = 0$ (Exercise). ,

Hence applying Itô's formula we obtain

$$\begin{aligned} d(e^{-rt}v(t, S(t), Y(t))) &= e^{-rt}(\partial_t v + rx\partial_x v + \frac{1}{2}\sigma^2 x^2 \partial_x^2 v - rv)(t, S(t), Y(t)) dt \\ &\quad + e^{-rt}\sigma S(t)\partial_x v(t, S(t), Y(t))d\widetilde{W}(t) + e^{-rt}\partial_y v(t, S(t), Y(t))dY(t). \end{aligned}$$

The drift term $(\dots) dt$ is zero by the PDE (1a). We now show that the term $(\dots)dY(t)$ is also zero, thereby concluding that $\{e^{-rt}v(t, S(t), Y(t))\}_{t \geq 0}$ is a $\widetilde{\mathbb{P}}$ -martingale relative to $\{\mathcal{F}_W(t)\}_{t \geq 0}$.

Since $Y(t)$ is non-increasing, then it has bounded first variation and therefore the integral

$$\int_0^t \partial_y v(\tau, S(\tau), Y(\tau))dY(\tau)$$

can be understood in the Riemann–Stieltjes sense.

We divide this integral as

$$\begin{aligned} \int_0^t \partial_y v(\tau, S(\tau), Y(\tau)) dY(\tau) &= \int_{S(\tau) > Y(\tau)} \partial_y v(\tau, S(\tau), Y(\tau)) dY(\tau) \\ &+ \int_{S(\tau) = Y(\tau)} \partial_y v(\tau, S(\tau), Y(\tau)) dY(\tau). \end{aligned}$$

The second piece is zero by the boundary condition (1b).

The second piece is also zero, because $\{S(\tau) > Y(\tau)\}$ is an open set (as S, Y are time-continuous) and $Y(\tau)$ is constant in this set (that is “ $dY(\tau) = 0$ ”).

It follows that the process $\{e^{-rt}v(t, S(t), Y(t))\}_{t \geq 0}$ is a $\tilde{\mathbb{P}}$ -martingale relative to $\{\mathcal{F}_W(t)\}_{t \geq 0}$; in particular

$$\tilde{\mathbb{E}}[e^{-rT}v(T, S(T), Y(T)) | \mathcal{F}_W(t)] = e^{-rt}v(t, S(t), Y(t)).$$

Hence, by the terminal condition (1c),

$$v(t, S(t), Y(t)) = e^{-r(T-t)} \tilde{\mathbb{E}}[S(T) - Y(T) | \mathcal{F}_W(t)],$$

which is the claim. □

To study the problem (1) one needs a complete set of boundary conditions for strong solutions.

Assume first that $y \rightarrow 0$. This means that the stock price has reached the value zero at some time $0 \leq \tau \leq t$, in which case of course the minimum stock price in the interval $[0, T]$ will be zero with probability 1.

Hence for $y \rightarrow 0^+$, the lookback call price converges to its highest possible value, that is

$$v(t, x, 0) = x. \quad t \in [0, T]. \tag{2}$$

The boundary value as $x \rightarrow \infty$ is not so obvious. In the next theorem we show that the 1+2 dimensional problem (1) can be reduced to a 1+1 dimensional problem with explicit boundary conditions.

Theorem 2. *Let $u : (1, \infty) \rightarrow \mathbb{R}$ satisfy*

$$\partial_t u + rz \partial_z u + \frac{1}{2} \sigma^2 z^2 \partial_{zz} u = ru, \quad t \in (0, T), \quad z > 1 \tag{3a}$$

with terminal condition

$$u(T, z) = z - 1 \tag{3b}$$

and boundary conditions

$$\lim_{z \rightarrow \infty} (u(t, z) - z) = 0, \quad u(t, 1) - \partial_z u(t, 1) = 0. \tag{3c}$$

Then

$$v(t, x, y) = yu\left(t, \frac{x}{y}\right)$$

solves (1) as well as (2).