Financial derivatives and PDE's
Lecture 18

Simone Calogero

February 17^{th}, 2021

Stochastic volatility models

For local volatility models, the stock price and the instantaneous volatility are both stochastic processes.

However there is only one source of randomness which drives both these processes, namely a single Brownian motion $\{W(t)\}_{t\geq 0}$.

The next level of generalization consists in assuming that the stock price and the volatility are driven by two different sources of randomness.

Definition 1. *Let $\{W_1(t)\}_{t\geq 0}$, $\{W_2(t)\}_{t\geq 0}$ be two independent Brownian motions and $\{\mathcal{F}_W(t)\}_{t\geq 0}$ be their own generated filtration. Let $\rho \in [-1,1]$ be a deterministic constant and $\mu, \eta, \beta : [0,\infty)^3 \to \mathbb{R}$ be continuous functions. A* ***stochastic volatility model*** *is a pair of (non-negative) stochastic diffusion processes $\{S(t)\}_{t\geq 0}$, $\{v(t)\}_{t\geq 0}$ satisfying the following system of SDE's:*

$$dS(t) = \mu(t, S(t), v(t))S(t)\,dt + \sqrt{v(t)}S(t)\,dW_1(t), \tag{1}$$

$$dv(t) = \eta(t, S(t), v(t))\,dt + \beta(t, S(t), v(t))\sqrt{v(t)}(\rho\,dW_1(t) + \sqrt{1-\rho^2}\,dW_2(t)). \tag{2}$$

We see from (1) that $\{v(t)\}_{t\geq 0}$ is the instantaneous variance of the stock price $\{S(t)\}_{t\geq 0}$.

Moreover the process $\{W^{(\rho)}(t)\}_{t\geq 0}$ given by

$$W^{(\rho)}(t) = \rho\,W_1(t) + \sqrt{1-\rho^2}\,W_2(t)$$

is a Brownian motion satisfying

$$dW_1(t)dW^{(\rho)}(t) = \rho\,dt;$$

in particular the two Brownian motions $\{W_1(t)\}_{t\geq 0}$, $\{W^{(\rho)}\}_{t\geq 0}$ are not independent, as their cross variation is not zero; in fact, by ρ is the correlation of the two Brownian motions.

Hence in a stochastic volatility model the stock price and the volatility are both stochastic processes driven by two correlated Brownian motions. We assume that $\{S(t)\}_{t\geq 0}$ is non-negative and $\{v(t)\}_{t\geq 0}$ is positive a.s. for all times, although we refrain from discussing under which general conditions this is verified (we will present an example below).

Our next purpose is to introduce a risk-neutral probability measure such that the discounted price of the stock is a martingale.

As we have two Brownian motions in this model, we shall apply the multi-dimensional Girsanov Theorem to construct such a probability measure.

Let

$$\{W_1(t)\}_{t\geq 0}, \dots, \{W_N(t)\}_{t\geq 0}$$

be independent Brownian motions and let $\{\mathcal{F}_W(t)\}_{t\geq 0}$ be their own generated filtration. Let $\{\theta_1(t)\}_{t\geq 0}, \dots, \{\theta_N(t)\}_{t\geq 0} \in \mathcal{C}^0[\mathcal{F}_W(t)]$ and set $\theta(t) = (\theta_1(t), \dots, \theta_N(t))$.

We assume that the Novikov condition (**??**) is satisfied (with $\theta(t)^2 = \|\theta(t)\|^2 = \theta_1(t)^2 + \cdots + \theta_N(t)^2$).

Then the stochastic process $\{Z(t)\}_{t\geq 0}$ given by

$$Z(t) = \exp\left(- \sum_{j=1}^{N} \int_0^t \theta_j(s) dW_j(s) - \frac{1}{2} \int_0^t \|\theta(s)\|^2 ds \right)$$

is a martingale relative to $\{\mathcal{F}_W(t)\}_{t\geq 0}$. It follows as before that the map $\widetilde{\mathbb{P}} : \mathcal{F} \to [0,1]$ given by

$$\widetilde{\mathbb{P}}(A) = \mathbb{E}[Z(T)\mathbb{I}_A], \quad A \in \mathcal{F} \tag{3}$$

is a new probability measure equivalent to $\mathbb{P}$ and the following N-dimensional generalization of Girsanov's theorem holds.

Theorem 1. *Define the stochastic processes* $\{\widetilde{W}_1(t)\}_{t\geq 0}, \dots, \{\widetilde{W}_N(t)\}_{t\geq 0}$ *by*

$$\widetilde{W}_k(t) = W_k(t) + \int_0^t \theta_k(s) ds, \quad k = 1, \dots, N. \tag{4}$$

Then $\{\widetilde{W}_1(t)\}_{t\geq 0}, \dots, \{\widetilde{W}_N(t)\}_{t\geq 0}$ *are independent Brownian motions in the probability measure* $\widetilde{\mathbb{P}}$. *Moreover the filtration* $\{\mathcal{F}_W(t)\}_{t\geq 0}$ *generated by* $\{W_1(t)\}_{t\geq 0}, \dots, \{W_N(t)\}_{t\geq 0}$ *is a non-anticipating filtration for* $\{\widetilde{W}_1(t)\}_{t\geq 0}, \dots, \{\widetilde{W}_N(t)\}_{t\geq 0}$.

Now let $r > 0$ be the constant interest rate of the money market and $\gamma : [0, \infty)^3 \to \mathbb{R}$ be a continuous function We define

$$\theta_1(t) = \frac{\mu(t, S(t), v(t)) - r}{\sqrt{v(t)}}, \quad \theta_2(t) = \gamma(t, S(t), v(t)), \quad \theta(t) = (\theta_1(t), \theta_2(t)).$$

Given $T > 0$, we introduce the new probability measure $\widetilde{\mathbb{P}}^{(\gamma)}$ equivalent to $\mathbb{P}$ by $\widetilde{\mathbb{P}}^{(\gamma)}(A) = \mathbb{E}[Z(T)\mathbb{I}_A]$, for all $A \in \mathcal{F}$, where

$$Z(t) = \exp\left(- \int_0^t \theta_1(s)\, dW_1(s) - \int_0^t \theta_2(s)\, dW_2(s) - \frac{1}{2} \int_0^t \|\theta(s)\|^2\, ds \right).$$

Then by Theorem 1, the stochastic processes

$$\widetilde{W}_1(t) = W_1(t) + \int_0^t \theta_1(s)\, ds, \quad \widetilde{W}_2^{(\gamma)}(t) = W_2(t) + \int_0^t \gamma(s)\, ds$$

are two $\widetilde{\mathbb{P}}^{(\gamma)}$-independent Brownian motions. Moreover (1)-(2) can be rewritten as

$$\begin{aligned} dS(t) &= rS(t)dt + \sqrt{v(t)}S(t)d\widetilde{W}_1(t), & \text{(5a)} \\ dv(t) &= [\eta(t, S(t), v(t)) - \sqrt{v(t)}\psi(t, S(t), v(t))\beta(t, S(t), v(t))]dt \\ &\quad + \beta(t, S(t), v(t))\sqrt{v(t)}d\widetilde{W}^{(\rho,\gamma)}, & \text{(5b)} \end{aligned}$$

where $\{\psi(t, S(t), v(t))\}_{t\geq 0}$ is the $\{\mathcal{F}_W(t)\}_{t\geq 0}$-adapted stochastic process given by

$$\psi(t, S(t), v(t)) = \frac{\mu(t, S(t), v(t)) - r}{\sqrt{v(t)}}\rho + \gamma(t, S(t), v(t))\sqrt{1 - \rho^2} \tag{6}$$

and where

$$\widetilde{W}^{(\rho,\gamma)}(t) = \rho\widetilde{W}_1(t) + \sqrt{1 - \rho^2}\widetilde{W}_2^{(\gamma)}(t).$$

Note that the $\widetilde{\mathbb{P}}^{(\gamma)}$-Brownian motions $\{\widetilde{W}_1(t)\}_{t\geq 0}$, $\{\widetilde{W}^{(\rho,\gamma)}(t)\}_{t\geq 0}$ satisfy

$$d\widetilde{W}_1(t)d\widetilde{W}^{(\rho,\gamma)}(t) = \rho dt, \qquad \text{for } \rho \in [-1, 1]. \tag{7}$$

It follows immediately that the discounted price $\{e^{-rt}S(t)\}_{t\geq 0}$ is a $\widetilde{\mathbb{P}}^{(\gamma)}$-martingale relative to the filtration $\{\mathcal{F}_W(t)\}_{t\geq 0}$. Hence *all* probability measures $\widetilde{\mathbb{P}}^{(\gamma)}$ are equivalent risk-neutral probability measures.

Remark (Incomplete markets).

As the risk-neutral probability measure is not uniquely defined, the market under discussion is said to be **incomplete**. Within incomplete markets there is no unique value for the price of derivatives (it depends on which specific risk-neutral probability measure is used to price the derivative). The stochastic process $\{\psi(t)\}_{t\geq 0}$ is called the **market price of volatility risk** and reduces to $\theta(t) = (\mu(t) - r(t))/\sigma(t)$ for $\gamma \equiv 0$ (or $\rho = 1$).

Consider now the standard European derivative with pay-off $Y = g(S(T))$ at time of maturity T.

For stochastic volatility models it is reasonable to assume that the risk-neutral price $\Pi_Y(t)$ of the derivative is a local function of the stock price *and* of the instantaneous variance, i.e., we make the following *ansatz* which generalizes (**??**):

$$\Pi_Y(t) = e^{-r(T-t)}\widetilde{\mathbb{E}}[g(S(T))|\mathcal{F}_W(t)] = v_g(t, S(t), v(t)) \tag{8}$$

for all $t \in [0,T]$, for all $T > 0$ and for some measurable pricing v_g.

As in the case of local volatility models, (8) is motivated by the Markov property of solutions to systems of SDE's.

In fact, it is useful to consider a more general European derivative with pay-off Y given by

$$Y = h(S(T), v(T)),$$

for some function $h : [0,\infty)^2 \to \mathbb{R}$, i.e., the pay-off of the derivative depends on the stock value *and* on the instantaneous variance of the stock at the time of maturity.

Theorem 2. *Assume that the functions $\eta(t,x,y)$, $\beta(t,x,y)$, $\psi(t,x,y)$ in (5) are such that the PDE*

$$\partial_t u + rx\partial_x u + A\partial_y u + \frac{1}{2}yx^2\partial_x^2 u + \frac{1}{2}\beta^2 y\partial_y^2 u + \rho\beta xy\partial_{xy}^2 u = 0, \tag{9a}$$

$$A = \eta - \sqrt{y}\beta\psi, \quad (t,x,y) \in (0,T) \times (0,\infty)^2 \tag{9b}$$

admits a unique strong solution u satisfying $u(T,x,y) = h(x,y)$. Then the risk-neutral price of the derivative with pay-off $Y = h(S(T), v(T))$ and maturity T is given by

$$\Pi_Y(t) = f_h(t, S(t), v(t))$$

where the pricing function f_h is given by $f_h(t,x,y) = e^{-r\tau}u(t,x,y)$, $\tau = T - t$.

As for the local volatility models, a closed formula solution of (9) is rarely available and the use numerical methods to price the derivative becomes essential.

Heston model

A popular stochastic volatility model is the **Heston** model, which is obtained by the following substitutions in (1)-(2):

$$\mu(t,S(t),v(t)) = \mu_0, \quad \beta(t,x,y) = c, \quad \eta(t,x,y) = a(b-y),$$

where μ_0, a, b, c are constant. Hence the stock price and the volatility dynamics in the Heston model are given by the following stochastic differential equations:

$$dS(t) = \mu_0 S(t)\,dt + \sqrt{v(t)}S(t)dW_1(t), \tag{10a}$$

$$dv(t) = a(b-v(t))dt + c\sqrt{v(t)}dW^{(\rho)}(t). \tag{10b}$$

Note in particular that the variance in the Heston model is a CIR process in the physical probability $\mathbb{P}$.

The condition $2ab \geq c^2$ ensures that $v(t)$ is strictly positive almost surely.

To pass to the risk neutral world we need to fix a risk-neutral probability measure, that is, we need to fix the market price of volatility risk function ψ in (6).

In the Heston model it is assumed that

$$\psi(t,x,y) = \lambda\sqrt{y}, \tag{11}$$

for some constant $\lambda \in \mathbb{R}$, which leads to the following form of the pricing PDE (9):

$$\partial_t u + rx\partial_x u + (k - my)\partial_y u + \frac{1}{2}yx^2\partial_x^2 u + \frac{c^2}{2}y\partial_y^2 u + \rho cxy\partial_{xy}^2 u = 0, \tag{12}$$

where the constant k, m are given by $k = ab$, $m = (a + c\lambda)$. Note that the choice (11) implies that the variance of the stock remains a CIR process in the risk-neutral probability measure.

The general solution of (12) with terminal datum $u(T,x,y) = h(x,y)$ is not known.

However in the case of a call option (i.e., $h(x,y) = g(x) = (x-K)_+$) an explicit formula for the Fourier transform of the solution is available, see [**?**].

With this formula at hand one can compute the price of call options by very efficient numerical methods, which is one of the main reasons for the popularity of the Heston model.

Variance swaps

Variance swaps are financial derivatives on the realized annual variance of an asset (or index).

We first describe how the realized annual variance is computed from the historical data of the asset price. Let $T > 0$ be measured in days and consider the partition

$$0 = t_0 < t_1 < \cdots < t_n = T, \quad t_{j+1} - t_j = h > 0,$$

of the interval $[0,T]$.

Assume for instance that the asset is a stock and let $S(t_j) = S_j$ be the stock price at time t_j.

Here $S_1, \ldots S_n$ are historical data for the stock price and *not* random variables (i.e., the interval $[0,T]$ lies in the past of the present time).

The **realized annual variance** of the stock in the interval $[0,T]$ along this partition is defined as

$$\sigma^2_{\text{1year}}(n,T) = \frac{\kappa}{T}\sum_{j=0}^{n-1}\left(\log\frac{S_{j+1}}{S_j} - \frac{1}{n}\log\frac{S(T)}{S(0)}\right)^2, \tag{13}$$

where κ is the number of trading days in one year (typically, $\kappa = 252$).

Using $T = nh$ we see that, up to a normalization factor, (13) coincides with the sample variance of the log-returns of the stock in the intervals $[t_j, t_{j+1}]$, $j = 0, \ldots, n-1$.

A **variance swap** stipulated at time $t = 0$, with maturity T and strike variance K is a contract between two parties which, at the expiration date, entails the exchange of cash given by $N(\sigma^2_{\text{1year}} - K)$, where N (called **variance notional**) is a conversion factor from units of variance to units of currency.

In particular, the holder of the long position on the swap is the party who receives the cash in the case that the realized annual variance at the expiration date is larger than the strike variance.

Variance swaps are traded over the counter and they are used by investors to protect their exposure to the volatility of the asset.

For instance, suppose that an investor has a position on an asset which is profitable if the volatility of the stock price increases (e.g., the investor owns call options on the stock).

Then it is clearly important for the investor to secure such position against a possible decrease of the volatility.

To this purpose the investor opens a short position on a variance swap with another investor who is exposed to the opposite risk.

Let us now discuss variance swaps from a mathematical modeling point of view.

We assume that the stock price follows the generalized geometric Brownian motion

$$S(t) = S(0) \exp\left(\int_0^t \alpha(s)ds + \int_0^t \sigma(s)dW(s)\right).$$

Next we show that, as $n \to \infty$, the realized annual variance in the future time interval $[0, T]$ converges in L^2 to the random variable

$$Q_T = \frac{\kappa}{T}[\log S, \log S](T) = \frac{\kappa}{T}\int_0^T \sigma^2(t)\,dt.$$

To see this we first rewrite the definition of realized annual variance as

$$\sigma^2_{1\text{year}}(n, T) = \frac{\kappa}{T}\sum_{j=0}^{n-1}\left(\log\frac{S_{j+1}}{S_j}\right)^2 - \frac{\kappa}{nT}\left(\log\frac{S(T)}{S(0)}\right)^2. \tag{14}$$

Hence

$$\lim_{n\to\infty}\sigma^2_{1\text{year}}(n, T) = \lim_{n\to\infty}\frac{\kappa}{T}\sum_{j=0}^{n-1}\left(\log\frac{S_{j+1}}{S_j}\right)^2 \quad \text{in } L^2.$$

Moreover, by the definition of quadratic variation, it follows that

$$\mathbb{E}\left[\left(\frac{\kappa}{T}\sum_{j=0}^{n-1}\left(\log\frac{S_{j+1}}{S_j}\right)^2 - Q_T\right)^2\right] \to 0, \quad \text{as } n \to \infty.$$

A variance swap can thus be defined as the (non-standard) European derivative with pay-off $Y = Q_T - K$.

Assuming that the interest rate of the money market is the constant $r \in \mathbb{R}$, the risk-neutral value of a variance swap is given by

$$\Pi_Y(t) = e^{-r\tau}\widetilde{\mathbb{E}}[Q_T - K|\mathcal{F}_W(t)]. \tag{15}$$

In particular, at time $t = 0$, i.e., when the contract is stipulated, we have

$$\Pi_Y(0) = e^{-rT}\widetilde{\mathbb{E}}[Q_T - K], \tag{16}$$

where we used that $\mathcal{F}_W(0)$ is a trivial σ-algebra, and therefore the conditional expectation with respect to $\mathcal{F}_W(0)$ is a pure expectation.

As none of the two parties in a variance swap has a privileged position on the contract, there is no premium associated to variance swaps, that is to say, the fair value of a variance swap is zero (his is a general property of forward contracts).

The value K_* of the variance strike which makes the risk-neutral price of a variance swap equal to zero at time $t = 0$, i.e., $\Pi_Y(0) = 0$, is called the **fair variance strike**.

By (16) we find

$$K_* = \frac{\kappa}{T}\int_0^T \widetilde{\mathbb{E}}[\sigma^2(t)]\,dt. \tag{17}$$

To compute K_* explicitly, we need to fix a stochastic model for the variance process $\{\sigma^2(t)\}_{t\geq 0}$.

Let us consider the Heston model

$$d\sigma^2(t) = a(b - \sigma^2(t))dt + c\sigma(t)d\widetilde{W}(t), \tag{18}$$

where a, b, c are positive constants satisfying $2ab \geq c^2$ and where $\{\widetilde{W}(t)\}_{t\geq 0}$ is a Brownian motion in the risk-neutral probability measure.

To compute the fair variance strike of the swap using the Heston model we use that

$$\widetilde{\mathbb{E}}[\sigma^2(t)] = abt - a\int_0^t \widetilde{\mathbb{E}}[\sigma^2(s)]\,ds,$$

which implies $\frac{d}{dt}\widetilde{\mathbb{E}}[\sigma^2(t)] = ab - a\widetilde{\mathbb{E}}[\sigma^2(t)]$ and so

$$\widetilde{\mathbb{E}}[\sigma^2(t)] = b + (\sigma_0^2 - b)e^{-at}, \quad \sigma_0^2 = \widetilde{\mathbb{E}}[\sigma^2(0)] = \sigma^2(0). \tag{19}$$

Replacing into (17) we obtain

$$K_* = \kappa\left[b + \frac{\sigma_0^2 - b}{aT}(1 - e^{-aT})\right].$$

Exercise 1. *Given* $\sigma_0 > 0$, *let* $\sigma(t) = \sigma_0 \sqrt{S(t)}$, *which is an example of CEV model. Compute the fair strike of the variance swap.*

Exercise 2. *Assume that the price $S(t)$ of a stock follows a generalized geometric Brownian motion with instantaneous volatility $\{\sigma(t)\}_{t\geq 0}$ given by the Heston model $d\sigma^2(t) = a(b - \sigma^2(t))\,dt + c\sigma(t)\,d\widetilde{W}(t)$, where $\{\widetilde{W}(t)\}_{t\geq 0}$ is a Brownian motion in the risk-neutral probability measure and a, b, c are constants such that $2ab \geq c^2 > 0$. The volatility call option with strike K and maturity T is the financial derivative with pay-off*

$$Y = N\left(\sqrt{\frac{\kappa}{T}\int_0^T \sigma^2(t)\,dt} - K\right)_+,$$

where κ is the number of trading days in one year and N is a dimensional constant that converts units of volatility into units of currency. Assuming that the interest rate of the money market is constant, find the partial differential equation and the terminal value satisfied by the pricing function of the volatility option.