Financial derivatives and PDE's
Lecture 19

Simone Calogero

February 18^{th}, 2021

Zero-coupon bonds

A **zero-coupon bond** (ZCB) with **face** (or **nominal**) value K and maturity $T > 0$ is a contract that promises to pay to its owner the amount K at time T in the future.

Zero-coupon bonds, and related contracts described below, are issued by national governments and private companies as a way to borrow money and fund their activities.

In the following we assume that all ZCB's are issued by one given institution, so that all bonds differ merely by their face values and maturities.

Moreover without loss of generality we assume from now on that $K = 1$, as owning a ZCB with face value K is clearly equivalent to own K shares of a ZCB with face value 1.

Once a debt is issued in the so-called **primary market**, it becomes a tradable asset in the **secondary** bond market.

It is therefore natural to model the value at time t of the ZCB maturing at time $T > t$ (and face value 1) as a random variable, which we denote by $B(t, T)$.

We assume throughout the discussion that the institution issuing the bond bears no risk of default, i.e., $B(t, T) > 0$, for all $t \in [0, T]$.

Clearly $B(T, T) = 1$ and, under normal market conditions, $B(t, T) < 1$, for $t < T$, although exceptions are not rare.

A **zero-coupon bond market** (ZCB market) is a market in which the objects of trading are ZCB's with different maturities. Our main goal is to introduce models for the prices of ZCB's observed in the market.

For modeling purposes we assume that zero-coupon bonds are available with a continuum range of maturities $T \in [0, S]$, where $S > 0$ is sufficiently large so that all ZCB's in the market mature before the time S (e.g., $S \approx 50$ years).

Mathematically this means that we model the prices of ZCB's in the market as a stochastic process depending on 2 parameters, namely

$$\{B(t,T), t \in [0,T], T \in [0,S]\}.$$

All processes $\{X(t,T), t \in [0,T], T \in [0,S]\}$ introduced in this section are assumed to have a.s. continuous paths in both variables t, T and to be adapted to the filtration $\{\mathcal{F}_W(t)\}_{t\geq 0}$ generated by the given Brownian motion $\{W(t)\}_{t\geq 0}$.

This means that for all *given* $T \in [0, S]$, the stochastic process $\{X(t,T)\}_{t\in[0,T]}$ is adapted to $\{\mathcal{F}_W(t)\}_{t\geq 0}$. By abuse of notation we continue to denote by $\mathcal{C}^0[\mathcal{F}_W(t)]$ this class of processes.

Forward and spot rate

The difference in value of zero-coupon bonds with different maturities is expressed through the implied forward rate of the bond.

To define this concept, suppose first that at the present time t we open a portfolio that consists of -1 share of a zero-coupon bond with maturity $T > t$ and $B(t,T)/B(t,T+\delta)$ shares of a zero-coupon bond expiring at time $T + \delta$.

Note that the value of this portfolio is $V(t) = 0$. This investment entails that we pay 1 at time T and receive $B(t,T)/B(t,T+\delta)$ at time $T + \delta$.

Hence our investment at time t is equivalent to an investment in the future time interval $[T, T+\delta]$ with (annualized) return given by

$$F_\delta(t,T) = \frac{1}{\delta}\left(B(t,T)/B(t,T+\delta) - 1\right) = \frac{B(t,T) - B(t,T+\delta)}{\delta B(t,T+\delta)}. \tag{1}$$

The quantity $F_\delta(t,T)$ is also called the **simply compounded forward rate** in the interval $[T, T+\delta]$ **locked at time** t (or forward LIBOR, as it is commonly applied to LIBOR interest rate contracts).

The name is intended to emphasize that the return in the future interval $[T, T+\delta]$ is locked at the time $t \leq T$, that is to say, we know today which interest rate has to be charged to borrow in the future time interval $[T, T+\delta]$ (if a different rate were locked today, then an arbitrage opportunity would arise).

In the limit $\delta \to 0^+$ we obtain the so called **continuously compounded T-forward rate** of the bond locked at time t:

$$F(t,T) = \lim_{\delta \to 0^+} F_\delta(t,T) = -\frac{1}{B(t,T)} \partial_T B(t,T) = -\partial_T \log B(t,T), \tag{2}$$

where $0 \leq t \leq T$ and $0 \leq T \leq S$.

Inverting (2) we obtain

$$B(t,T) = \exp\left(-\int_t^T F(t,v)\, dv \right), \quad 0 \leq t \leq T \leq S. \tag{3}$$

By (3), a model for the price $B(t,T)$ of the ZCB's in the market can be obtained by a model on the **forward rate curve** $T \to F(t,T)$. This approach to the problem of ZCB pricing is known as HJM approach, from Heath, Jarrow, Morton, who introduced this method in the late 1980s.

Letting $T \to t^+$ in (1) we obtain the **simply compounded spot rate**,

$$R_\delta(t) = \lim_{T \to t^+} F_\delta(t,T), \tag{4}$$

that is to say, the interest rate locked "on the spot", i.e., at the present time t, to borrow in the interval $[t, t+\delta]$. Letting $\delta \to 0^+$ we obtain the **instantaneous** (or **continuously compounded**) **spot rate** $\{r(t)\}_{t \in [0,S]}$ of the ZCB market:

$$r(t) = \lim_{\delta \to 0^+} R_\delta(t) = \lim_{T \to t^+} F(t,T), \quad t \in [0,S]. \tag{5}$$

Note that $r(t)$ is the interest rate locked at time t to borrow in the "infinitesimal interval" of time $[t, t+dt]$. Hence $r(t)$ coincides with the risk-free rate of the money market used in the previous sections.

For the options pricing problem studied ibefore we assumed that $r(t)$ was equal to a constant r, which is reasonable for short maturity contracts ($T \lesssim 1$ year).

However when large maturity assets such as ZCB's are considered, we have to relax this assumption and promote $\{r(t)\}_{t \in [0,S]}$ to a stochastic process.

In the so-called **classical approach** to the problem of ZCB's pricing, the fair value of $B(t,T)$ is derived from a model on the spot rate process.

Yield to maturity of ZCB's

If a ZCB is bought at time t and kept until its maturity $T > t$, the annualized log-return of the investment is

$$Y(t,T) = -\frac{1}{T-t} \log B(t,T) = \frac{1}{T-t} \int_t^T F(t,v)\, dv, \tag{6}$$

which is called the **(continuously compounded) yield to maturity** of the zero-coupon bond, while $T \to Y(t,T)$ is called the **yield curve**.

Inverting (6) we find

$$B(t,T) = e^{-Y(t,T)(T-t)} = \frac{D_Y(T)}{D_Y(t)}, \quad \text{where } D_Y(s) = e^{-Y(t,T)s}. \tag{7}$$

Hence we may interpret the yield also as the *constant interest rate which entails that the value of the ZCB at time t equals the discounted value of the future payment 1 at time T.*

Coupon bonds

Let $0 < t_1 < t_2 < \cdots < t_M = T$ be a partition of the interval $[0,T]$.

A **coupon bond** with maturity T, face value 1 and coupons $c_1, c_2, \ldots, c_M \in [0,1)$ is a contract that promises to pay the amount c_k at time t_k and the amount $1 + c_M$ at maturity $T = t_M$.

Note that some c_k may be zero, which means that no coupon is actually paid at that time.

We set $c = (c_1, \ldots, c_M)$ and denote by $B_c(t,T)$ the value at time t of the bond paying the coupons $c_1, \ldots, c_M$ and maturing at time T.

Now, let $t \in [0,T]$ and $k(t) \in \{1, \ldots, M\}$ be the smallest index such that $t_{k(t)} > t$, that is to say, $t_{k(t)}$ is the first time after t at which a coupon is paid.

Holding the coupon bond at time t is clearly equivalent to holding a portfolio containing $c_{k(t)}$ shares of the ZCB expiring at time $t_{k(t)}$, $c_{k(t)+1}$ shares of the ZCB expiring at time $t_{k(t)+1}$, and so on, hence

$$B_c(t,T) = \sum_{j=k(t)}^{M-1} c_j B(t,t_j) + (1+c_M)B(t,T), \tag{8}$$

the sum being zero when $k(t) = M$.

The yield to maturity of a coupon bond is the quantity $Y_c(t,T)$ defined implicitly by the equation

$$B_c(t,T) = \sum_{j=k(t)}^{M-1} c_j e^{-Y_c(t,T)(t_j-t)} + (1+c_M)e^{-Y_c(t,T)(T-t)}. \tag{9}$$

It follows that *the yield of the coupon bond is the constant interest rate used to discount the total future payments of the coupon bond.*

Example. Consider a 3 year maturity coupon bond with face value 1 which pays 2% coupon semiannually. Suppose that the bond is listed with an yield of 1%. What is the value of the bond at time zero? The coupon dates are

$$(t_1, t_2, t_3, t_4, t_5, t_6) = (1/2, 1, 3/2, 2, 5/2, 3),$$

and $c_1 = c_2 = \cdots = c_6 = 0.02$. Hence

$$\begin{aligned} B_c(0,T) &= 0.02e^{-0.01*\frac{1}{2}} + 0.02e^{-0.01*1} + 0.02e^{-0.01*\frac{3}{2}} + 0.02e^{-0.01*2} + 0.02e^{-0.01*\frac{5}{2}} \\ &\quad + (1+0.02)e^{-0.01*3} = 1.08837. \end{aligned}$$

Remark. As in the previous example, the coupons of a coupon bond are typically all equal, i.e., $c_1 = c_2 = \cdots = c_M = c \in (0,1)$.

In the example above, the yield was given and $B_c(0,T)$ was computed.

However one is most commonly faced with the opposite problem, i.e., computing the yield of the coupon bond with given initial value $B_c(0,T)$.

We can easily solve this problem numerically inverting (9).

For instance, assume that the bond is issued at time $t = 0$ with maturity $T = M$ years (M integer) and that the coupons are paid annually, that is $t_1 = 1$, $t_2 = 2, \ldots, t_M = M$.

Then $x = \exp(-Y_c(0,T))$ solves $p(x) = 0$, where p is the M-order polynomial given by

$$p(x) = c_1 x + c_2 x^2 + \cdots + (1 + c_M) x^M - B_c(0,T). \tag{10}$$

The roots of this polynomial can easily be computed numerically.

Yield curve

The curve $T \to y_c(t,T)$ is called the **yield curve** of the bond market at time t.

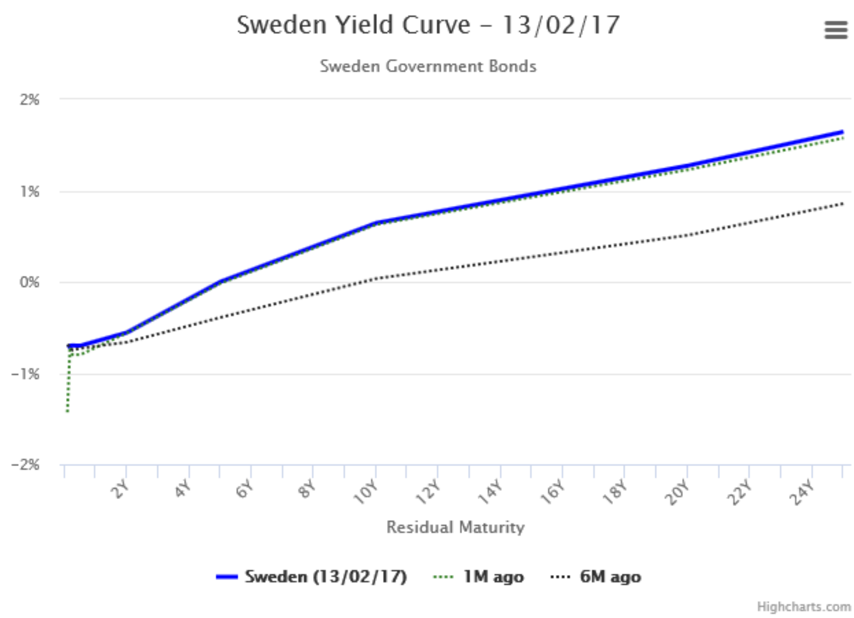

Figure 1: Yield curve for Swedish bonds. Note that the yield is negative for maturities shorter than 5 years.

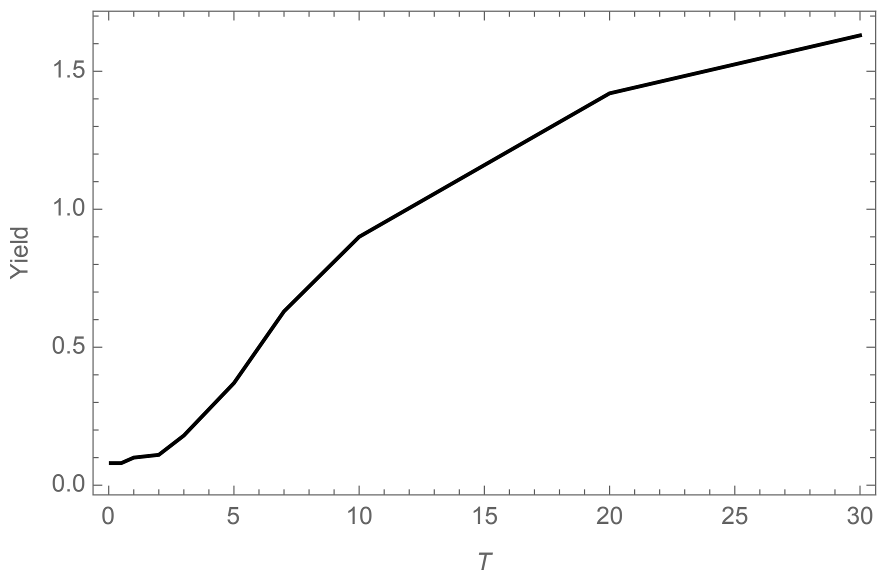

Figure 2: Yield curve for US bonds (10 December 2020).