Lecture 16

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Financial derivatives and PDE's
Lecture 16

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1 A project on the Asian option

The Asian call option[1] with strike $K > 0$ and maturity $T > 0$ is the non-standard European derivative with pay-off $Y = \left(\frac{1}{T}\int_0^T S(t)\,dt - K\right)_+$ and similarly one defines the Asian put option. The Black-Scholes price of the Asian option can be computed numerically either by the Monte Carlo method or by solving a boundary value problem

$$\partial_t u + \frac{\sigma^2}{2}(\gamma(t) - z)^2 \partial_z^2 u = 0, \quad t \in (0, T), z \in \mathbb{R} \tag{1a}$$

$$u(T, z) = (z)_+, \quad \lim_{z\to-\infty} u(t, z) = 0, \; \lim_{z\to\infty}(u(t, z) - z) = 0, \quad t \in [0, T), \tag{1b}$$

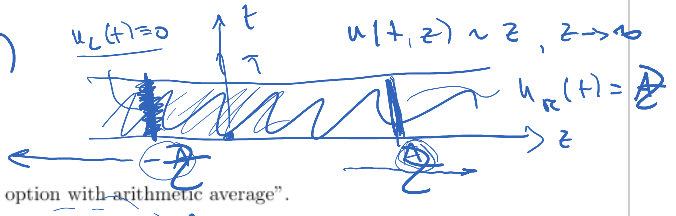
using the finite difference method. The main purpose of this project is to compare the performance of the two methods. For the finite difference approach it is convenient to invert the time direction in the boundary value problem by changing variable $t \to T - t$, thereby obtaining the system

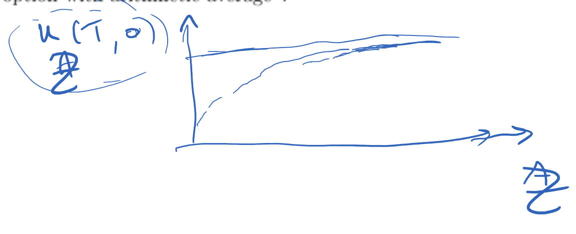
$$-\partial_t u + \frac{\sigma^2}{2}(\gamma(t) - z)^2 \partial_z^2 u = 0, \quad t \in (0, T), z \in \mathbb{R} \tag{2a}$$

$$u(0, z) = z^+, \quad \lim_{z\to-\infty} u(t, z) = 0, \; \lim_{z\to\infty}(u(t, z) - z) = 0, \quad t \in (0, T], \tag{2b}$$

$(z)_+ = \max(z, 0)$

where $\gamma(t) = \frac{1-e^{-rt}}{rT}$.





[1] In this project "Asian option" always means "Asian option with arithmetic average".

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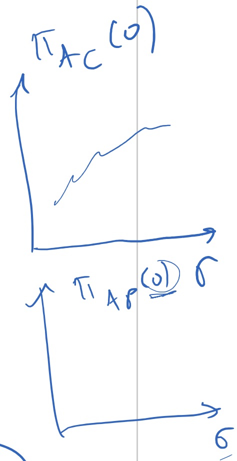
Part 1

- Write a short introduction on the Asian call/put option, where you should discuss in particular its financial utility and main differences with the standard European call/put (you can find plenty of info on the web). Outline the content of the rest of the report.
- Solve Exercise 6.18.
- Write a finite difference scheme that solves the PDE (2) in the domain $(t, z) \in (0, T) \times (-Z, Z)$ with the appropriate initial and boundary conditions for Asian calls or puts. Use the Crank-Nicolson method.

Part 2

- Write a Matlab function that implements the finite difference scheme derived in Part 1. The parameters S_0, r, σ, K, T, must appear as input variables of your function.
- Plot the initial price of the Asian call/put as a function of the volatility σ and of the initial price S_0. Discuss your findings, in particular the relation with the behavior of standard call/put options. Verify numerically the validity of the put-call parity.
- Compare the price obtained by the finite difference method and by the control variate Monte Carlo method for different value of σ and compare the efficiency of the two methods, e.g., by performing speed tests. Present your results using tables and discuss them.



Include your matlab codes in an appendix with a description of how they work (e.g., as comments within the codes themselves).

	MC ≈ 100		
MONTE CARLO	5.5 s		
FINITE DIFFERENCE	0.5 s		

2 A project on the CEV model

In the CEV (Constant Elasticity Variance) model, the price of the stock option with maturity $T > 0$ and pay-off $Y = g(S(T))$ is given by $\Pi_Y(t) = e^{-r(T-t)}u(t, S(t))$, where $S(t)$ is the stock price at time t and u solves

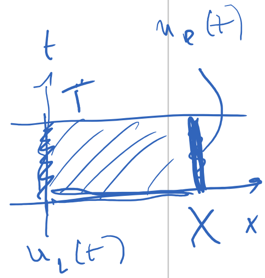
$$\partial_t u + rx\partial_x u + \frac{\sigma^2}{2}x^{2\delta}\partial_x^2 u = 0, \ x > 0, \ t \in (0, T), \tag{3}$$

with the terminal condition $u(T, x) = g(x)$, $x > 0$. Here $r > 0$, $\sigma > 0$, $\delta > 0$ are constants; see Section **??** for a derivation of the model. Actually it is more convenient to work with the equivalent problem

$$-\partial_t u + rx\partial_x u + \frac{\sigma^2}{2}x^{2\delta}\partial_x^2 u = 0, \ x > 0, \ t \in (0, T), \tag{4a}$$

$$u(0, x) = g(x), \quad x > 0 \tag{4b}$$

which is obtained by the change of variable $t \to T - t$ in (3).



The main purpose of this project is to derive numerically some qualitative properties of the CEV model, such as the implied volatility curve.

Part 1

- Write a short introduction on the CEV model, where you should discuss in particular the applications of the CEV model (you can find plenty of info on the web) and the main differences with the Black-Scholes model. Outline the content of the rest of the report.
- Solve Exercise ??. 6.27
- Write a finite difference scheme for the PDE (4a) on the domain $(t, x) \in (0, T) \times (0, X)$ with initial data and boundary conditions corresponding to European call or put options. Use the Crank-Nicolson method (see Remark **??**).

Part 2

- Write a Matlab function that implements the finite difference scheme derived in Part 1 in the case of call/put options. The parameters $S_0, r, K, \sigma, \delta, X, T$, must appear as input variables of your function.
- Compare the finite difference solution for $\delta = 1$ with the exact Black-Scholes solution and discuss possible sources of error and how to eliminate them.

- Plot the initial price of call and put options as a function of the initial stock price and of the volatility parameter σ. Verify numerically the validity of the put-call parity. Highlight the main differences between the Black-Scholes price and the CEV model price of call/put options.
- Compare the price obtained by the finite difference method and the Monte Carlo method for different values of σ and compare the efficiency of the two methods, e.g., by performing speed tests. Present your results using tables and discuss them. Remark: Apply the Euler-Maruyama method to generate paths of the stock price.

Include your matlab codes in an appendix with a description of how they work (e.g., as comments within the codes themselves)

$$dS(t) = rS(t) + \sigma S(t)^{\delta} d\widetilde{W}(t)$$

Exercise 6.18

The Asian call with **geometric average** is the European style derivative with pay-off

$$Z = \left(\exp\left(\frac{1}{T}\int_0^T \log S(t)\,dt\right) - K\right)_+,$$

where $T > 0$ and $K > 0$ are respectively the maturity and strike of the call. Derive an exact formula for the Black-Scholes price of this option and for the corresponding put option. Derive also the put-call parity. Prove that the Asian call with geometric average is cheaper than the corresponding Asian call with arithmetic average.

SAY AT $t = 0$

ANSWER:

$$\Pi_{AC}^{(G)}(0) = e^{-rT}\left(e^{qT} S_0 \Phi(d_1) - K\Phi(d_2)\right)$$

$$q = \frac{1}{2}\left(r - \frac{\sigma^2}{6}\right) \cancel{\times}, \quad d_2 = d_1 - \sigma\sqrt{\frac{T}{3}}$$

$$d_1 = \frac{\log\frac{S_0}{K} + \frac{1}{2}\left(r + \frac{\sigma^2}{6}\right)T}{\sigma\sqrt{T/3}}$$

Exercise 6.27

Given σ, r and $\delta \neq 1$, define

$$a = 2r(\delta - 1),\ \ c = -2\sigma(\delta - 1),\ \ b = \frac{\sigma^2}{2r}(2\delta - 1),\ \ \theta = -\frac{1}{2(\delta - 1)}.$$

Let $\{X(t)\}_{t\geq 0}$ be the CIR process

$$dX(t) = a(b - X(t))\,dt + c\sqrt{X(t)}d\widetilde{W}(t), \quad X(0) = x > 0.$$

Show that $S(t) = X(t)^\theta$ solves

$$dS(t) = rS(t)dt + \sigma S(t)^\delta d\widetilde{W}(t), \quad S(0) = S_0 > 0. \tag{5}$$

with $S_0 = x^\theta$.