Lecture 17

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Financial derivatives and PDE's
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Simone Calogero

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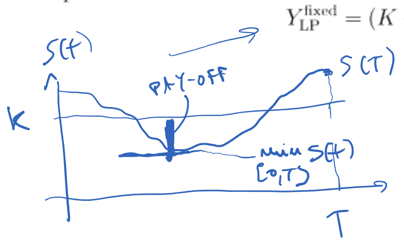
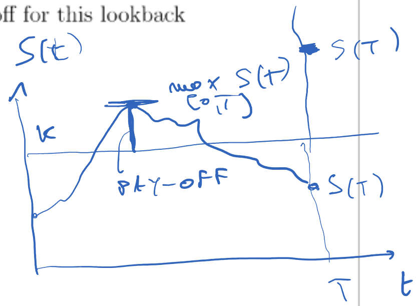
The lookback option

Lookback options are non-standard European style derivatives whose pay-off depends on the minimum or maximum of the stock price within a given time period until maturity. There exists four main types of lookback options.

- A lookback **call** option with **floating strike** and maturity $T > 0$ gives to the owner the right to buy the underlying stock at maturity for the minimum price of the stock in the interval $[0, T]$. Thus the pay-off for this lookback option is
$$Y_{\mathrm{LC}}^{\mathrm{float}} = S(T) - \min\{S(t), t \in [0, T]\}. \geq 0 \quad (= 0 \text{ IFF } \min S(t) = S(T))$$
- A lookback **put** option with **floating strike** and maturity $T > 0$ gives to the owner the right to sell the underlying stock at maturity for the maximum price of the stock in the interval $[0, T]$. Thus the pay-off for this lookback option is
$$Y_{\mathrm{LP}}^{\mathrm{float}} = \max\{S(t), t \in [0, T]\} - S(T). \geq 0$$
- A lookback **call** option with **fixed strike** $K > 0$ and maturity $T > 0$ pays the buyer the difference between the maximum of the stock price in the interval $[0, T]$ and the strike K, provided this difference is positive. Hence the pay-off for this lookback option is
$$Y_{\mathrm{LC}}^{\mathrm{fixed}} = (\max\{S(t), t \in [0, T]\} - K)_+. \geq (S(T) - K)_+$$
- A lookback **put** option with **fixed strike** $K > 0$ and maturity $T > 0$ pays the buyer the difference between the strike price K and the minimum of the stock price in the interval $[0, T]$, provided this difference is positive. Hence the pay-off for this lookback option is
$$Y_{\mathrm{LP}}^{\mathrm{fixed}} = (K - \min\{S(t), t \in [0, T]\})_+.$$



$\geq (K - S(T))_+$



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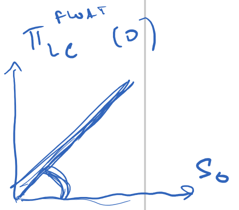
In the following section we focus on the lookback call option with floating strike in a Black-Scholes market. In particular the stock price is given by

$$S(t) = S(0)e^{(r-\frac{1}{2}\sigma^2)t+\sigma\widetilde{W}(t)}$$

and the risk-neutral price for the floating strike lookback call option with maturity $T > 0$ is

$\Pi_{LC}^{FLOAT}(T) = \underbrace{S(T)}_{x} - \underbrace{\min_{[0,T]} S(t)}_{y}$

$$\Pi_{\mathrm{LC}}^{\mathrm{float}}(t) = e^{-r(T-t)}\widetilde{\mathbb{E}}[\underbrace{S(T)}_{x} - \underbrace{\min\{S(\tau), \tau \in [0,T]\}}_{y}|\mathcal{F}_W(t)].$$



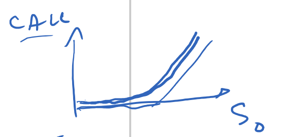
Exercise 1. *Show that the price at time $t = 0$ of the lookback call and put options with floating strike is a linear increasing function of the stock price at time zero.*

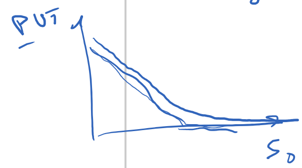
SOLUTION: $\Pi_{LC}^{FLOAT}(t) = e^{-r(T-t)}\tilde{E}\Big[S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma\tilde{W}(T)} - \min_{[0,T]} S_0 e^{(r-\frac{1}{2}\sigma^2)\tau+\sigma\tilde{W}(\tau)}\Big]$

$= S_0 \big[e^{-r(T-t)}\tilde{E}[\dots]\big] = a S_0$

Pricing PDE for the lookback call option with floating strike

$\Delta = \partial_x S$





The main purpose of this section is to derive the PDE satisfied by the pricing function of lookback call options with floating strike.

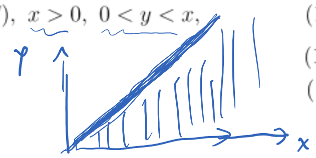
Theorem 1. *Let $v : (0,T) \times (0,\infty) \times (0,\infty)$, $v = v(t,x,y)$, satisfy*

$$\partial_t v + rx\partial_x v + \frac{1}{2}\sigma^2 x^2 \partial_x^2 v = rv, \quad t \in (0,T),\ x > 0,\ 0 < y < x, \tag{1a}$$

NEUMANN BOUNDARY CONDITION

$$\partial_y v(t,x,x) = 0, \quad t \in [0,T], x > 0, \tag{1b}$$

$$v(T,x,y) = x - y, \quad 0 \le y \le x. \tag{1c}$$



Then $\Pi_{\mathrm{LC}}^{\mathrm{float}}(t) = v(t, \underbrace{S(t)}_{x}, \underbrace{\min_{0\le\tau\le t} S(\tau)}_{y \le x})$.

Proof. Let $Y(t) = \min_{0\le\tau\le t} S(\tau)$; note that $\{Y(t)\}_{t\ge0}$ is a non-increasing process in the space $\mathcal{C}^0[\mathcal{F}_W(t)]$. However $\{Y(t)\}_{t\ge0}$ is *not* a diffusion process.

We now show that

$$\boxed{dY(t)dY(t) = 0.} \quad (*)$$

Recall that this means that the quadratic variation of $\{Y(t)\}_{t\ge0}$ is zero in any interval $[0,T]$ along any sequence of partitions $\{\Pi_n\}_{n\in\mathbb{N}}$ of this interval such that $\|\Pi_n\| \to 0$, as $n \to \infty$.

Letting $\Pi_n = \{t_0 = 0, t_1^{(n)}, \dots, t_{m(n)}^n = T\}$, we have to prove that

$$\mathbb{E}[Q_M^2] \to 0 \qquad M \to \infty$$

$$(*) \iff \quad \lim_{n\to\infty} \sum_{j=1}^{m(n)} (Y(t_j^{(n)}) - Y(t_{j-1}^{(n)}))^2 = 0 \quad \text{in } L^2(\Omega).$$

But

$$-(Y(t_j^{(n)}) - Y(t_{j-1}^{(n)}))$$

$$\begin{aligned}
Q_M \le \sum_{j=1}^{m(n)} (Y(t_j^{(n)}) - Y(t_{j-1}^{(n)}))^2 &\le \max_j |Y(t_j^{(n)}) - Y(t_{j-1}^{(n)})| \sum_j |Y(t_j^{(n)}) - Y(t_{j-1}^{(n)})| \\
&= \max_j |Y(t_{j-1}^{(n)}) - Y(t_j^{(n)})| \sum_j (Y(t_{j-1}^{(n)}) - Y(t_j^{(n)})) \\
&= \max_j |Y(t_{j-1}^{(n)}) - Y(t_j^{(n)})| (Y(0) - Y(T)),
\end{aligned}$$

$$\xrightarrow[M\to\infty]{} 0 \quad \text{POINTWISE FOR ALL } \omega \in \Omega$$

where in the sum we used that $Y(t)$ is non-increasing to write $|Y(t) - Y(s)| = Y(s) - Y(t)$, for $t \ge s$.

As Y is continous in time, then $\max_j |Y(t_{j-1}^{(n)}, \omega) - Y(t_j^{(n)}, \omega)| \to 0$, pointwise in $\omega \in \Omega$.

As $Y(t_j^{(n)}) \le Y(0) = S_0$, then, by the dominated convergence theorem, the limit is zero also in L^2, which completes the proof of $dY(t)dY(t) = 0$.

$$\mathbb{E}[Q_M^2] \to 0 \qquad M \to \infty$$

Similarly one can prove that $dS(t)dY(t) = 0$ (Exercise).

$$Y(t) = \min_{0 \le \tau \le t} S(t)$$

Hence applying Itô's formula we obtain

$$\begin{aligned}
(*) \quad d(e^{-rt}v(t, S(t), Y(t))) &= e^{-rt}(\partial_t v + rx\partial_x v + \frac{1}{2}\sigma^2 x^2 \partial_x^2 v - rv)(t, S(t), Y(t))\, dt \\
&\quad + e^{-rt}\sigma S(t)\partial_x v(t, S(t), Y(t)) d\widetilde{W}(t) + e^{-rt}\partial_y v(t, S(t), Y(t)) dY(t).
\end{aligned}$$

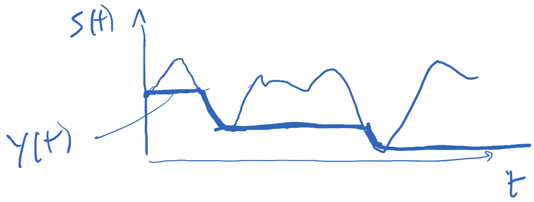
The drift term $(\dots)\,dt$ is zero by the PDE (1a). We now show that the term $(\dots)dY(t)$ is also zero, thereby concluding that $\{e^{-rt}v(t, S(t)Y(t)\}_{t\ge0}$ is a $\widetilde{\mathbb{P}}$-martingale relative to $\{\mathcal{F}_W(t)\}_{t\ge0}$.

Since $Y(t)$ is non-increasing, then it has bounded first variation and therefore the integral

$$0 = \int_0^t \partial_y v(\tau, S(\tau), Y(\tau)) dY(\tau)$$

can be understood in the Riemann–Stieltjes sense.

We divide this integral as



$$\int_0^t \partial_y v(\tau, S(\tau), Y(\tau))dY(\tau) = \int_{S(\tau)>Y(\tau)} \partial_y v(\tau, S(\tau), Y(\tau))dY(\tau)$$
$$+ \int_{S(\tau)=Y(\tau)} \partial_y v(\tau, S(\tau), Y(\tau))dY(\tau).$$

0

The second piece is zero by the boundary condition (1b).

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The ~~second~~ piece is also zero, because $\{S(\tau) > Y(\tau)\}$ is an open set (as S, Y are time-continuous) and $Y(\tau)$ is constant in this set (that is "$dY(\tau) = 0$").

It follows that the process $\{e^{-rt}v(t, S(t)Y(t)\}_{t\geq 0}$ is a $\widetilde{\mathbb{P}}$-martingale relative to $\{\mathcal{F}_W(t)\}_{t\geq 0}$; in particular

$S(T) - Y(T)$

$$\widetilde{\mathbb{E}}[e^{-rT}v(T, S(T), Y(T))|\mathcal{F}_W(t)] = e^{-rt}v(t, S(t), Y(t)).$$

$T > t$

Hence, by the terminal condition (1c),

$$v(t, S(t), Y(t)) = e^{-r(T-t)}\widetilde{\mathbb{E}}[S(T) - Y(T)]|\mathcal{F}_W(t)], \quad = \Pi^{\text{FLOAT}}_{\text{LC}}(t)$$

which is the claim. □

To study the problem (1) one needs a complete set of boundary conditions for strong solutions.

Assume first that $y \to 0$. This means that the stock price has reached the value zero at some time $0 \leq \tau \leq t$, in which case of course the minimum stock price in the interval $[0, T]$ will be zero with probability 1.

Hence for $y \to 0^+$, the lookback call price converges to its highest possible value, that is

$$v(t, x, 0) = x. \quad t \in [0, T]. \tag{2}$$

The boundary value as $x \to \infty$ is not so obvious. In the next theorem we show that the 1+2 dimensional problem (1) can be reduced to a 1+1 dimensional problem with explicit boundary conditions.

$(0,\infty)\times$ $\qquad u(t,z)$

Theorem 2. *Let* $u : (1, \infty) \to \mathbb{R}$ *satisfy*

$$\partial_t u + rz\partial_z u + \frac{1}{2}\sigma^2 z^2 \partial_{zz} u = ru, \ t \in (0, T), \ z > 1 \tag{3a}$$

with terminal condition

$$u(T,z) = z - 1 \tag{3b}$$

and boundary conditions

$$\lim_{z\to\infty} (u(t,z) - z) = 0, \quad u(t,1) - \partial_z u(t,1) = 0. \tag{3c}$$

Then

$$\boxed{v(t,x,y) = yu\left(t, \frac{x}{y}\right)}$$

solves (1) *as well as* (2).