Financial derivatives and PDE's
Lecture 23

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Futures

Futures contracts are standardized forward contracts, i.e., rather than being traded over the counter, they are negotiated in regularized markets.

Perhaps the most interesting role of futures contracts is that they make trading on commodities possible for anyone.

To this regard we remark that commodities, e.g. crude oil, wheat, etc, are most often sold through long term contracts, such as forward and futures contracts, and therefore they do not usually have an "official spot price", but only a future delivery price (commodities "spot markets" exist, but their role is marginal for the discussion in this section).

Futures markets are markets in which the objects of trading are futures contracts. Unlike forward contracts, all futures contracts in a futures market are subject to the same regulation, and so in particular all contracts on the same asset with the same delivery time T have the same delivery price, which is called the T-**future price** of the asset and which we denote by $\mathrm{Fut}_T(t)$. Thus $\mathrm{Fut}_T(t)$ is the delivery price in a futures contract on the asset with time of delivery T and which is stipulated at time $t < T$.

Futures markets have been existing for more than 300 years and nowadays the most important ones are the Chicago Mercantile Exchange (CME), the New York Mercantile Exchange (NYMEX), the Chicago Board of Trade (CBOT) and the International Exchange Group (ICE).

In a futures market, anyone (after a proper authorization) can stipulate a futures contract.

More precisely, holding a position in a futures contract in the futures market consists in the agreement to receive as a cash flow the change in the future price of the underlying asset

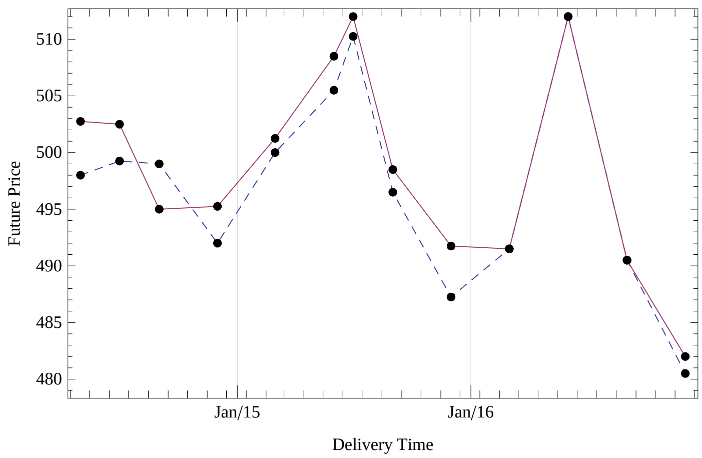

Figure 1: Futures price of corn on May 12, 2014 (dashed line) and on May 13, 2014 (continuous line) for different delivery times

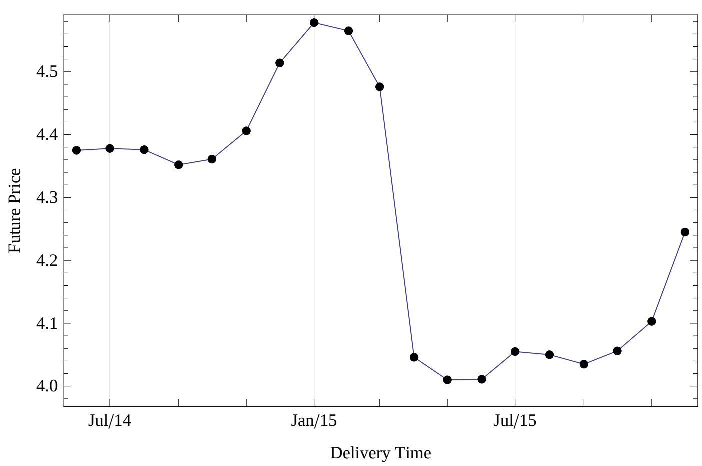

Figure 2: Futures price of natural gas on May 13, 2014 for different delivery times

during the time in which the position is held. Note that the cash flow may be positive or negative.

In a long position the cash flow is positive when the future price goes up and it is negative when the future price goes down , while a short position on the same contract receives the opposite cash flow.

Moreover, in order to eliminate the risk of insolvency, the cash flow is distributed in time through the mechanism of the **margin account**.

More precisely, assume that at $t = 0$ we open a long position in a futures contract expiring at time T.

At the same time, we need to open a margin account which contains a certain amount of cash (usually, 10 % of the current value of the T-future price for each contract opened).

At $t = 1$ day, the amount $\mathrm{Fut}_T(1) - \mathrm{Fut}_T(0)$ will be added to the account, if it positive, or withdrawn, if it is negative.

The position can be closed at any time $t < T$ (multiple of days), in which case the total amount of cash flown in the margin account is

$$(\mathrm{Fut}_T(t) - \mathrm{Fut}_T(t-1)) + (\mathrm{Fut}_T(t-1) - \mathrm{Fut}_T(t-2)) + \\ \cdots + (\mathrm{Fut}_T(1) - \mathrm{Fut}_T(0)) = (\mathrm{Fut}_T(t) - \mathrm{Fut}_T(0)).$$

If a long position is held up to the time of maturity, then the holder of the long position should buy the underlying asset.

Our next purpose is to derive a mathematical model for the future price of an asset.

Our guiding principle is that the **1+1 dimensional futures market** consisting of a futures contract and a risk-free asset should not admit self-financing arbitrage portfolios.

Consider a portfolio invested in $h(t)$ shares of the futures contract and $h_B(t)$ shares of the risk-free asset at time t.

We assume that $\{h(t), h_B(t)\}_{t\in[0,T]}$ is adapted to $\{\mathcal{F}_W(t)\}_{t\geq 0}$ and suppose that $\{\mathrm{Fut}_T(t)\}_{t\in[0,T]}$ is a diffusion process.

Since futures contracts have zero-value, the value of the portfolio at time t is $V(t) = h_B(t)B(t)$.

For a self-financing portfolio we require that any positive cash-flow generated by the futures contract in the interval $[t, t + dt]$ should be invested to buy shares of the risk-free asset and that, conversely, any negative cash flow should be settled by issuing shares of the risk-free asset (i.e., by borrowing money).

Since the cash-flow generated in the interval $[t, t+dt]$ is given by $dC(t) = h(t)d\mathrm{Fut}_T(t)$, the value of a self-financing portfolio invested in the 1+1 dimensional futures market must satisfy

$$dV(t) = h_B(t)dB(t) + h(t)d\mathrm{Fut}_T(t) = r(t)V(t)dt + h(t)d\mathrm{Fut}_T(t),$$

or equivalently

$$dV^*(t) = h(t)D(t)d\mathrm{Fut}_T(t). \tag{1}$$

Now, we have seen that a simple condition ensuring that a portfolio is not an arbitrage is that its discounted value be a martingale in the risk-neutral measure relative to the filtration generated by the Brownian motion.

By (1), the latter condition is achieved by requiring that $d\mathrm{Fut}_T(t) = \Delta(t)d\widetilde{W}(t)$, for some stochastic process $\{\Delta(t)\}_{t\in[0,T]}$ adapted to $\{\mathcal{F}_W(t)\}_{t\in[0,T]}$.

In particular, it is reasonable to impose that

(i) $\{\mathrm{Fut}_T(t)\}_{t\in[0,T]}$ should be a $\widetilde{\mathbb{P}}$-martingale relative to $\{\mathcal{F}_W(t)\}_{t\geq 0}$.

Furthermore, it is clear that the future price of an asset at the expiration date T should be equal to its spot price at time T, and so we impose that

(ii) $\mathrm{Fut}_T(T) = \Pi(T)$.

It follows by Exercise **??** that the conditions **(i)-(ii)** determine a unique stochastic process $\{\mathrm{Fut}_T(t)\}_{t\in[0,T]}$, which is given in the following definition.

Definition 1. *Assume that the price $\{\Pi(t)\}_{t\geq 0}$ of the asset satisfies*

$$d\Pi(t) = \alpha(t)\Pi(t)dt + \sigma(t)\Pi(t)dW(t),$$

*where $\{\alpha(t)\}_{t\geq 0}, \{\sigma(t)\}_{t\geq 0}, \{r(t)\}_{t\geq 0} \in \mathcal{C}^0[\mathcal{F}_W(t)]$ and $\sigma(t) > 0$ almost surely for all times. The T-**Future price** at time t of the asset is the $\{\mathcal{F}_W(t)\}_{t\geq 0}$-adapted stochastic process $\{\mathrm{Fut}_T(t)\}_{t\in[0,T]}$ given by*

$$\mathrm{Fut}_T(t) = \widetilde{\mathbb{E}}[\Pi(T)|\mathcal{F}_W(t)], \quad t \in [0,T].$$

We now show that our goal to make the futures market arbitrage-free has been achieved.

Theorem 1. *There exists a stochastic process* $\{\Delta(t)\}_{t\in[0,T]}$ *adapted to* $\{\mathcal{F}_W(t)\}_{t\geq 0}$ *such that*

$$\mathrm{Fut}_T(t) = \mathrm{Fut}_T(0) + \int_0^t \Delta(s)d\widetilde{W}(s). \tag{2}$$

Moreover, any $\{\mathcal{F}_W(t)\}_{t\geq 0}$*-adapted self-financing portfolio* $\{h(t), h_B(t)\}_{t\in[0,T]}$ *invested in the 1+1 dimensional futures market is not an arbitrage.*

Proof. The second statement follows immediately by the first one, since (1) and (2) imply that the value of a self-financing portfolio invested in the 1+1 dimensional futures market is a $\widetilde{\mathbb{P}}$-martingale relative to the filtration $\{\mathcal{F}_W(t)\}_{t\in[0,T]}$. To prove (2), we first notice that, by (**??**),

$$Z(s)\widetilde{\mathbb{E}}[\mathrm{Fut}_T(t)|\mathcal{F}_W(s)] = \mathbb{E}[Z(t)\mathrm{Fut}_T(t)|\mathcal{F}_W(s)].$$

By the martingale property of the future price, the left hand side is $Z(s)\mathrm{Fut}_T(s)$. Hence

$$Z(s)\mathrm{Fut}_T(s) = \mathbb{E}[Z(t)\mathrm{Fut}_T(t)|\mathcal{F}_W(s)],$$

that is to say, the process $\{Z(t)\mathrm{Fut}_T(t)\}_{t\in[0,T]}$ is a $\mathbb{P}$-martingale relative to the filtration $\{\mathcal{F}_W(t)\}_{t\in[0,T]}$. By the martingale representation theorem, Theorem **??**(v), there exists a stochastic process $\{\Gamma(t)\}_{t\in[0,T]}$ adapted to $\{\mathcal{F}_W(t)\}_{t\geq 0}$ such that

$$Z(t)\mathrm{Fut}_T(t) = \mathrm{Fut}_T(0) + \int_0^t \Gamma(s)dW(s).$$

We now proceed as in the proof of Theorem **??**, namely we write

$$d\mathrm{Fut}_T(t) = d(Z(t)\mathrm{Fut}_T(t)Z(t)^{-1})$$

and apply Itô's product rule and Itô's formula to derive that (2) holds with

$$\Delta(t) = \theta(t)\mathrm{Fut}_T(t) + \frac{\Gamma(t)}{Z(t)}.$$

□

Exercise 1. *Show that the* **Forward-Future spread** *of an asset, i.e., the difference between its forward and future price, satisfies*

$$\mathrm{For}_T(t) - \mathrm{Fut}_T(t) = \frac{1}{\widetilde{\mathbb{E}}[D(T)|\mathcal{F}_W(t)]}\Big\{\widetilde{\mathbb{E}}[D(T)\Pi(T)|\mathcal{F}_W(t)] - \widetilde{\mathbb{E}}[D(T)|\mathcal{F}_W(t)]\widetilde{\mathbb{E}}[\Pi(T)|\mathcal{F}_W(t)]\Big\}. \tag{3}$$

Moreover, show that when the interest rate $\{r(t)\}_{t\in[0,T]}$ *is a deterministic function of time (e.g., a deterministic constant), then* $\mathrm{For}_T(t) = \mathrm{Fut}_T(t)$*, for all* $t \in [0, T]$.