Financial derivatives and PDE's
Lecture 24

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February 26^{th}, 2021

1 Multi-dimensional markets

In this section we consider $N+1$ dimensional stock markets. We denote the stocks prices by

$$\{S_1(t)\}_{t\geq 0}, \dots, \{S_N(t)\}_{t\geq 0}$$

and assume the following dynamics

$$dS_k(t) = \left(\mu_k(t)\,dt + \sum_{j=1}^{N} \sigma_{kj}(t)dW_j(t)\right) S_k(t), \tag{1}$$

for some stochastic processes $\{\mu_k(t)\}_{t\geq 0}$, $\{\sigma_{kj}(t)\}_{t\geq 0}$, $j,k = 1,\dots,N$ in the class $\mathcal{C}^0[\mathcal{F}_W(t)]$, where in this section $\{\mathcal{F}_W(t)\}_{t\geq 0}$ denotes the filtration generated by the Brownian motions $\{W_1(t)\}_{t\geq 0}, \dots \{W_N(t)\}_{t\geq 0}$.

Moreover we assume that the Brownian motions are independent, in particular

$$dW_j(t)dW_k(t) = 0, \quad \text{for all } j \neq k \tag{2}$$

Finally $\{r(t)\}_{t\geq 0} \in \mathcal{C}^0[\mathcal{F}_W(t)]$ is the interest rate of the money market.

Now, given stochastic processes $\{\theta_k(t)\}_{t\geq 0} \in \mathcal{C}^0[\mathcal{F}_W(t)]$, $k = 1,\dots,N$, satisfying the Novikov condition (**??**), the stochastic process $\{Z(t)\}_{t\geq 0}$ given by

$$Z(t) = \exp\left(-\sum_{k=1}^{N}\left(\int_0^t \frac{1}{2}\theta_k^2(s)\,ds + \int_0^t \theta_k(s)\,dW_k(s)\right)\right) \tag{3}$$

is a martingale relative to the filtration $\{\mathcal{F}_W(t)\}_{t\geq 0}$.

Since $\mathbb{E}[Z(t)] = \mathbb{E}[Z(0)] = 1$, for all $t \geq 0$, we can use the stochastic process $\{Z(t)\}_{t\geq 0}$ to define a risk-neutral probability measure associated to the $N+1$ dimensional stock market, as we did in the one dimensional case.

Definition 1. *Let $T > 0$ and assume that the* **market price of risk equations**

$$\mu_j(t) - r(t) = \sum_{k=1}^{N} \sigma_{jk}(t)\theta_k(t), \quad j = 1, \ldots, N, \tag{4}$$

admit a solution $(\theta_1(t), \ldots, \theta_N(t))$, for all $t \geq 0$. Define the stochastic process $\{Z(t)\}_{t\geq 0}$ as in (3)*. Then the measure $\widetilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ given by*

$$\widetilde{\mathbb{P}}(A) = \mathbb{E}[Z(T)\mathbb{I}_A]$$

is called the risk-neutral probability measure of the market at time T.

Note that, as opposed to the one dimensional case, the risk-neutral measure just defined need not be unique, as the market price of risk equations may admit more than one solution.

For each risk-neutral probability measure $\widetilde{\mathbb{P}}$ we can apply the multidimensional Girsanov theorem and conclude that the stochastic processes $\{\widetilde{W}_1(t)\}_{t\geq 0}, \ldots \{\widetilde{W}_N(t)\}_{t\geq 0}$ given by

$$\widetilde{W}_k(t) = W_k(t) + \int_0^t \theta_k(s)\,ds$$

are $\widetilde{\mathbb{P}}$-independent Brownian motions. Moreover these Brownian motions are $\widetilde{\mathbb{P}}$-martingales relative to the filtration $\{\mathcal{F}_W(t)\}_{t\geq 0}$.

Now let $\{h_{S_1}(t)\}_{t\geq 0}, \ldots, \{h_{S_N}(t)\}_{t\geq 0} \in \mathcal{C}^0[\mathcal{F}_W(t)]$ be stochastic processes representing the number of shares on the stocks in a portfolio invested in the $N+1$ dimensional stock market.

Let $\{h_B(t)\}_{t\geq 0}$ be the number of shares on the risk-free asset. The portfolio value is

$$V(t) = \sum_{k=1}^{N} h_{S_k}(t)S_k(t) + h_B(t)B(t)$$

and the portfolio process is self-financing if its value satisfies

$$dV(t) = \sum_{k=1}^{N} h_{S_k}(t)dS_k(t) + h_B(t)dB(t),$$

that is

$$dV(t) = \sum_{k=1}^{N} h_{S_k}(t)dS_k(t) + r(t)\left(V(t) - \sum_{k=1}^{N} h_{S_k}(t)S_k(t)\right)dt.$$

Theorem 1. *Assume that a risk-neutral probability $\widetilde{\mathbb{P}}$ exists, i.e., the equations* (4) *admit a solution. Then the discounted value of any self-financing portfolio invested in the $N+1$ dimensional market is a $\widetilde{\mathbb{P}}$-martingale relative to the filtration $\{\mathcal{F}_W(t)\}_{t\geq 0}$. In particular (by Theorem* **??***) there exists no self-financing arbitrage portfolio invested in the $N+1$ dimensional stock market.*

Proof. The discounted value of the portfolio satisfies

$$\begin{aligned} dV^*(t) &= D(t)\left(\sum_{j=1}^{N} h_{S_j}(t)S_j(t)(\alpha_j(t) - r(t))\,dt + \sum_{j,k=1}^{N} h_{S_j}(t)S_j(t)\sigma_{jk}(t)dW_k(t)\right) \\ &= D(t)\left(\sum_{j=1}^{N} h_{S_j}(t)S_j(t)\sum_{k=1}^{N}\sigma_{jk}(t)\theta_k(t)dt + \sum_{j,k=1}^{N} h_{S_j}(t)S_j(t)\sigma_{jk}(t)dW_k(t)\right) \\ &= D(t)\sum_{j=1}^{N} h_{S_j}(t)S_j(t)\sum_{k=1}^{N}\sigma_{jk}(t)d\widetilde{W}_k(t). \end{aligned}$$

All Itô's integrals in the last line are $\widetilde{\mathbb{P}}$-martingales relative to $\{\mathcal{F}_W(t)\}_{t\geq 0}$. The result follows. □

Next we show that the existence of a risk-neutral probability measure is necessary for the absence of self-financing arbitrage portfolios in $N+1$ dimensional stock markets.

Let $N = 3$ and assume that the market parameters are constant.

Let $r(t) = r > 0$, $(\mu_1, \mu_2, \mu_3) = (2, 3, 2)$ and let the volatility matrix be given by

$$\sigma_{ij} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 1 & 2 & 0 \end{pmatrix}.$$

Thus the stocks prices satisfy

$$
\begin{aligned}
dS_1(t) &= (2dt + dW_1(t) + 2dW_2(t))S_1(t),\\
dS_2(t) &= (3dt + 2dW_1(t) + 4dW_2(t))S_2(t),\\
dS_3(t) &= (2dt + dW_1(t) + 2dW_2(t))S_3(t).
\end{aligned}
$$

The market price of risk equations are

$$
\begin{aligned}
\theta_1 + 2\theta_2 &= 2 - r\\
2\theta_1 + 4\theta_2 &= 3 - r\\
\theta_1 + 2\theta_2 &= 2 - r.
\end{aligned}
$$

This system is solvable if and only if $r = 1$, in which case there exist infinitely many solutions given by

$$
\theta_1 \in \mathbb{R}, \quad \theta_2 = \frac{1}{2}(1 - \theta_1).
$$

Hence for $r = 1$ there exists at least one (in fact, infinitely many) risk-neutral probability measures, and thus the market is free of arbitrage.

To construct an arbitrage portfolio when $0 < r < 1$, let

$$
h_{S_1}(t) = \frac{1}{S_1(t)}, \quad h_{S_2}(t) = -\frac{1}{S_2(t)}, \quad h_{S_3}(t) = \frac{1}{S_3(t)}
$$

and choose $h_B(t)$ such that the portfolio process is self-financing (see Exercise **??**).

The value $\{V(t)\}_{t\geq 0}$ of this portfolio satisfies

$$
\begin{aligned}
dV(t) &= h_{S_1}(t)dS_1(t) + h_{S_2}(t)dS_2(t) + h_{S_3}(t)dS_3(t)\\
&\quad + r(V(t) - h_{S_1}(t)S_1(t) - h_{S_2}(t)S_2(t) - h_{S_3}(t)S_3(t))dt\\
&= rV(t)dt + (1 - r)dt.
\end{aligned}
$$

Hence

$$
V(t) = V(0)e^{rt} + \frac{1}{r}(1 - r)(e^{rt} - 1)
$$

and this portfolio is an arbitrage, because for $V(0) = 0$ we have $V(t) > 0$, for all $t > 0$. Similarly one can find an arbitrage portfolio for $r > 1$.

Next we address the question of **completeness** of $N+1$ dimensional stock markets, i.e., the question of whether any European derivative can be hedged in this market.

Consider a European derivative on the stocks with pay-off Y and time of maturity T.

For instance, for a standard European derivative, $Y = g(S_1(T), \dots S_N(T))$, for some measurable function g.

The risk-neutral price of the derivative is

$$\Pi_Y(t) = \widetilde{\mathbb{E}}[Y \exp(-\int_t^T r(s)\,ds)|\mathcal{F}_W(t)],$$

and coincides with the value at time t of any self-financing portfolio invested in the $N+1$ dimensional market. The question of existence of an hedging portfolio is answered by the following theorem.

Theorem 2. *Assume that the volatility matrix $(\sigma_{jk}(t))_{j,k=1,\dots N}$ is invertible, for all $t \geq 0$. There exist stochastic processes $\{\Delta_1(t)\}_{t\in[0,T]}, \dots \{\Delta_N(t)\}_{t\in[0,T]}$, adapted to $\{\mathcal{F}_W(t)\}_{t\geq 0}$, such that*

$$D(t)\Pi_Y(t) = \Pi_Y(0) + \sum_{k=1}^{N} \int_0^t \Delta_k(s) d\widetilde{W}_k(s), \quad t \in [0,T]. \tag{5}$$

Let $(Y_1(t), \dots, Y_N(t))$ be the solution of

$$\sum_{k=1}^{N} \sigma_{jk}(t) Y_j(t) = \frac{\Delta_k(t)}{D(t)}. \tag{6}$$

Then the portfolio $\{h_{S_1}(t), \dots, h_{S_N}(t), h_B(t)\}_{t\in[0,T]}$ given by

$$h_{S_j}(t) = \frac{Y_j(t)}{S_j(t)}, \quad h_B(t) = (\Pi_Y(t) - \sum_{j=1}^{N} h_{S_j}(t) S_j(t))/B(t) \tag{7}$$

is self-financing and replicates the derivative at any time, i.e., its value $V(t)$ is equal to $\Pi_Y(t)$ for all $t \in [0,T]$. In particular, $V(T) = \Pi_Y(T) = Y$, i.e., the portfolio is hedging the derivative.

The proof of this theorem is conceptually very similar to the one dimensional case and is therefore omitted (it makes use of the multidimensional version of the martingale representation theorem).

Notice that, having assumed that the volatility matrix is invertible, *the risk-neutral probability measure of the market is unique.*

We now show that the uniqueness of the risk-neutral probability measure is necessary to guarantee completeness.

In fact, let $r = 1$ in the example considered before and pick the following solutions of the market price of risk equations:

$$(\theta_1, \theta_2) = (0, 1/2), \quad \text{and} \quad (\theta_1, \theta_2) = (1, 0)$$

(any other pair of solutions would work).

The two corresponding risk-neutral probability measures, denoted respectively by $\widetilde{\mathbb{P}}$ and $\widehat{\mathbb{P}}$, are given by

$$\widetilde{\mathbb{P}}(A) = \mathbb{E}[\widetilde{Z}\mathbb{I}_A] \quad \widehat{\mathbb{P}}(A) = \mathbb{E}[\widehat{Z}\mathbb{I}_A], \quad \text{for all } A \in \mathcal{F},$$

where

$$\widetilde{Z} = e^{-\frac{1}{8}T - \frac{1}{2}W_2(T)}, \quad \widehat{Z} = e^{-\frac{1}{2}T - W_1(T)}.$$

Let $A = \{\omega : \frac{1}{2}W_2(T,\omega) - W_1(T,\omega) < \frac{3}{8}T\}$. Hence

$$\widehat{Z}(\omega) < \widetilde{Z}(\omega), \quad \text{for } \omega \in A$$

and thus $\widehat{\mathbb{P}}(A) < \widetilde{\mathbb{P}}(A)$.

Consider a financial derivative with pay-off $Q = \mathbb{I}_A / D(T)$. If there existed an hedging, self-financing portfolio for such derivative, then, since the discounted value of such portfolio is a martingale in both risk-neutral probability measures, we would have

$$V(0) = \widetilde{\mathbb{E}}[QD(T)], \quad \text{and} \quad V(0) = \widehat{\mathbb{E}}[QD(T)]. \tag{8}$$

But

$$\widehat{\mathbb{E}}[QD(T)] = \widehat{\mathbb{E}}(\mathbb{I}_A) = \widehat{\mathbb{P}}(A) < \widetilde{\mathbb{P}}(A) = \widetilde{\mathbb{E}}(\mathbb{I}_A) = \widetilde{\mathbb{E}}[QD(T)]$$

and thus (8) cannot be verified.