Lecture 20

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Financial derivatives and PDE's
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$dr(t) = (a - b\,r(t))\,dt + c\sqrt{r(t)}\,d\widetilde{W}(t)$

$a = z$

Classical approach to ZCB pricing

In the so-called classical approach to the problem of pricing ZCB's we interpret the ZCB as a derivative on the spot rate.

Assume that a model for $\{r(t)\}_{t\in[0,S]}$ is given as a stochastic process in the space $\mathcal{C}^0[\mathcal{F}_W(t)]$. As the pay-off of the ZCB equals one, the risk-neutral price of the ZCB is given by

$B(0,T) = \widetilde{\mathbb{E}}[D(T)]$

$1 = Y$

$$B(t,T) = \widetilde{\mathbb{E}}[D(t)^{-1}D(T)|\mathcal{F}_W(t)] = \widetilde{\mathbb{E}}\Big[\exp\Big(-\int_t^T r(s)\,ds\Big)\Big|\mathcal{F}_W(t)\Big] \qquad (1)$$

and thus the discounted price of the ZCB, $B^*(t,T) = D(t)B(t,T)$, is a $\widetilde{\mathbb{P}}$-martingale relative to $\{\mathcal{F}_W(t)\}_{t\geq 0}$; in particular, self-financing portfolios invested in the ZCB market are not arbitrage portfolios.

Note however that *the risk-neutral probability measure in* (1) *cannot be determined solely by the spot rate, and therefore models for the process* $\{r(t)\}_{t\in[0,S]}$ *must be given a priori in terms of a risk-neutral probability measure.* As the real world is not risk-neutral, the foundation of the classical approach is questionable.

There are two ways to get around this problem. One is the HJM approach described below; the other is by adding a risky asset (e.g. a stock) to the ZCB market, which would then be used to determine the risk-neutral probability measure. The last procedure is referred to as "completing the ZCB market" and will be discussed in Section ??.

As an application of the classical approach, assume that the spot rate is given by the Cox-Ingersoll-Ross (CIR) model,

$$dr(t) = a(b - r(t))dt + c\sqrt{r(t)}\,d\widetilde{W}(t), \quad r(0) = r_0 > 0, \tag{2}$$

where $\{\widetilde{W}(t)\}_{t\in[0,T]}$ is a Brownian motion *in the risk-neutral probability measure* and R_0, a, b, c are positive constants.

To compute $B(t,T)$ under a CIR interest rate model, we make the *ansatz*

$v(T, r(T)) = 1$

$$B(t,T) = v(t, r(t)), \tag{3}$$

for some smooth function $v : [0,T] \times \mathbb{R} \to \mathbb{R}$, which we want to find. Note that we do not require the Feller condition $ab \geq c^2/2$, hence we allow the spot rate to become zero with positive probability, although negative values are excluded in the CIR model.

Theorem 1. *When the interest rate $\{r(t)\}_{t>0}$ follows the CIR model (2), the value $B(t,T)$ of the zero-coupon bond is given by*

$$v(t,x) = e^{-xC(\tau)-A(\tau)}, \quad \tau = T - t, \tag{4}$$

$B(t,T) = e^{-r(t)C(T-t) - A(T-t)}$

where $C(\tau)$, $A(\tau)$ satisfy the Cauchy problem

$$C'(\tau) = 1 - aC(\tau) - \frac{c^2}{2}C(\tau)^2, \quad A'(\tau) = abC(\tau) \tag{5a}$$

$$C(0) = 0, \quad A(0) = 0. \tag{5b}$$

Moreover the solution of the Cauchy problem (5) is given by

$$C(\tau) = \frac{\sinh(\gamma\tau)}{\gamma\cosh(\gamma\tau) + \frac{1}{2}a\sinh(\gamma\tau)} \tag{6a}$$

$$A(\tau) = -\frac{2ab}{c^2}\log\left[\frac{\gamma e^{\frac{1}{2}a\tau}}{\gamma\cosh(\gamma\tau) + \frac{1}{2}a\sinh(\gamma\tau)}\right] \tag{6b}$$

$D(t) = e^{-\int_0^t r(s)ds}$

$dD(t) = -r(t)D(t)dt$

and

$$\gamma = \frac{1}{2}\sqrt{a^2 + 2c^2}. \tag{6c}$$

Proof. Using Itô's formula and the product rule, together with (2), we obtain

$d(D(t)v(t, r(t)))$

$= (---)dt + (--)d\widetilde{W}(t)$

$$\begin{aligned} d(D(t)v(t,r(t))) = D(t)\Big[&\partial_t v(t,r(t)) + a(b - r(t))\partial_x v(t,r(t)) \\ &+ \frac{c^2}{2}r(t)\partial_x^2 v(t,r(t)) - r(t)v(t,r(t))\Big]dt \\ &+ D(t)\partial_x v(t,r(t))c\sqrt{r(t)}\,d\widetilde{W}(t). \end{aligned}$$

$(dD(t))v + (dv)D(t) + \cancel{dD\,dv}$

$dv = \partial_t v\,dt + (\partial_x v)\,dr(t) + \frac{1}{2}(\partial_x^2 v)\,dr\,dr$

$\partial_t v = v\,(x\,C'(\tau) + A'(\tau))$

$\partial_x v = v\,(-C(\tau)) \qquad \partial_x v = v\,C(\tau)^2$

Hence, imposing that v be a solution of the PDE

$x > 0$

$$\partial_t v + a(b - x)\partial_x v + \frac{c^2}{2}x\partial_x^2 v = xv, \quad (t,x) \in \mathcal{D}_T^+, \tag{7a}$$

we obtain that the stochastic process $\{D(t)v(t,r(t))\}_{t\in[0,T]}$ is a $\widetilde{\mathbb{P}}$-martingale relative to the filtration $\{\mathcal{F}_W(t)\}_{t\in[0,T]}$. Imposing additionally the terminal condition

$$v(T,x) = 1, \quad \text{for all } x > 0, \tag{7b}$$

we obtain

$$D(t)v(t,r(t)) = \widetilde{\mathbb{E}}[D(T)v(T,R(T))|\mathcal{F}_W(t)] = \widetilde{\mathbb{E}}[D(T)|\mathcal{F}_W(t)],$$

hence

$$v(t,r(t)) = \widetilde{\mathbb{E}}[D(T)/D(t)|\mathcal{F}_W(t)], \quad = B(t,T)$$

$v(t,x) = e^{-xC(\tau) - A(\tau)}$

and thus (3) is verified. Replacing the *ansatz* (4) into (7) we find the following first order polynomial equation

$$\boxed{x\Big(C'(\tau) + aC(\tau) + \frac{c^2}{2}C(\tau)^2 - 1\Big) + A'(\tau) - abC(\tau) = 0.} \quad \forall\ x > 0$$

$\tau = T - t$

The previous equation holds for all x if and only if (5a) hold, while the initial conditions (5b)

$$\boxed{x(C'(\tau) + aC(\tau) + \frac{c^2}{2}C(\tau)^2 - 1) + A'(\tau) - abC(\tau) = 0.} \quad \forall\, x > 0$$

$\tau = T - t$

The previous equation holds for all x if and only if (5a) hold, while the initial conditions (5b) are equivalent to the terminal condition $v(T, x) = 1$. The proof of the claim that (6) is the solution of the Cauchy problem (5) is left as an exercise. □

The CIR model is an example of **affine model**, i.e., a model for the interest rate which entails a price function for the ZCB of the form $B(t,T) = \exp(-r(t)C(t) - A(t))$ (or equivalently, an yield which is a linear function of the spot rate).

The most general affine model has the form

$$dr(t) = a(t)(b(t) - r(t))\,dt + c(t)\sqrt{r(t) + \delta(t)}\,d\widetilde{W}(t), \qquad (8)$$

where a, b, c, δ are deterministic functions of time.

$\sqrt{\gamma(t)\, r(t) + \delta(t)}$

Exercise 1. *Let $B(t,T) = v(t, r(t))$ the price of the ZCB with face value 1 entailed by the general affine model* (8). *Set* $v(t,x) = \exp(-xC(\underset{t,T}{\cancel{T-t}}) - A(\underset{t,T}{\cancel{T-t}}))$ *and derive the ODE's verified by the functions A, C.*

Exercise 2. *Assume that the interest rate of a zero-coupon bond is given by the Vasicek model*

$$dr(t) = a(b - r(t))dt + c\,d\widetilde{W}(t), \quad r(0) = r_0 \in \mathbb{R},$$

where a, b, c are positive constants and $\{\widetilde{W}(t)\}_{t\geq 0}$ is a Brownian motion in the risk-neutral probability measure $\widetilde{\mathbb{P}}$. Show that $r(t)$ is $\widetilde{\mathbb{P}}$-normally distributed and compute its expectation and variance in the risk-neutral probability measure. (Derive the PDE for the pricing function v of the ZCB with face value 1 and maturity $T > 0$. Find v using the ansatz (4).)

SOLUTION:

$$B(t,T) = \widetilde{\mathbb{E}}\left[e^{-\int_t^T r(s)\,ds} \mid \mathcal{F}_W(t) \right] = v(t, r(t))$$

$$D(t) = \exp\left(-\int_0^t r(s)\,ds\right) \Rightarrow dD(t) = -D(t)\,r(t)\,dt$$

$$d\big(D(t)v(t,r(t))\big) = \big(dD(t)\big)v(t,r(t)) + D(t)\,dv(t,r(t)) + \underbrace{dD\,dv}_{0}$$

$$= -D(t)r(t)dt\,v(t,r(t)) + D(t)\Big(\partial_t v\,dt + \partial_x v\,dr(t) + \frac{1}{2}\partial_x^2 v\,dr(t)dr(t)\Big)$$

$$= D(t)\Big[-r(t)v\,dt + \partial_t v\,dt + \partial_x v\big(a(b-r(t))dt + c\,d\widetilde{W}(t)\big) + \frac{1}{2}\partial_x^2 v\,c^2\,dt\Big]$$

$$= D(t)\Big[-rv + \partial_t v + a(b-x)\partial_x v + \frac{1}{2}c^2\partial_x^2 v\Big]_{x=r(t)} dt + c\,\partial_x v(t,r(t))\,D(t)\,d\widetilde{W}(t)$$

PDE:
$$\begin{cases} \partial_t v + a(b-x)\partial_x v + \frac{1}{2}c^2\partial_x^2 v = x\,v & t>0 \quad x\in\mathbb{R} \\ v(T,x) = 1 & x\in\mathbb{R} \end{cases}$$

$N(t,x) = e^{-xC(T-t) - A(T-t)}$ $\quad \partial_t N = N(xC'(\tau) + A'(\tau))$

$\partial_x N = N(-C)$ $\quad \partial^2_x N = N C^2$ $\Rightarrow$ PDE BECOMES

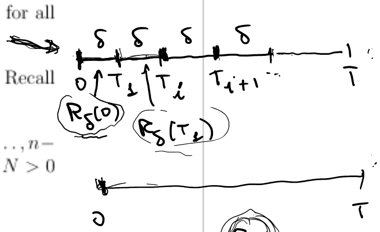
$N(xC' + A' - a(b-x)C + \frac{1}{2}c^2 C^2 - x) = 0$ FOR ALL x $\Rightarrow$ $C' = -aC + 1$

$A' = abC + \frac{1}{2}c^2 C^2$

Interest rate swap

An **interest rate swap** can be seen as a coupon bond with variable (random) coupons, which can be positive or negative.

More precisely, consider a partition $0 = T_0 < T_1 < \cdots < T_n = T$ with $T_i - T_{i-1} = \delta$, for all $i = 1, \ldots, n$.



Let $R_\delta(T_i) = F_\delta(T_i, T_i)$ be the simply compounded spot rate in the interval $[T_i, T_{i+1}]$. Recall that this quantity is known at time T_i (but not at time $t = 0$).

An interest rate swap is a contract between two parties which at each time T_{i+1}, $i = 1, \ldots, n-1$, entails the exchange of cash $N(R_\delta(T_i)\delta - r\delta)$, where r is a fixed interest rate and $N > 0$ is the notional amount converting units of interest rates into units of currency.

r

$N R_\delta(T_1)\delta$

$N r \delta$

Without loss of generality, we assume $N = 1$ in the following.

The party that receives this cash flow when it is positive is called the receiver, while the opposite party is called the payer.

Hence the receiver has a long position on the spot rate, while the payer has a short position on the spot rate.

The risk-neutral value at time $t = 0$ of the interest rate swap is the expectation, in the risk-neutral probability measure, of the discounted cash-flow entailed by the contract, that is

$$\Pi_{\mathrm{irs}}(0) = \delta \sum_{i=1}^{n-1} \widetilde{\mathbb{E}}[(\underbrace{R_\delta(T_i)}_{F_\delta(T_i,T_i)} - r)D(T_{i+1})], \quad = 0 \tag{9}$$

where $D(t)$ is the discount process.

Being a forward type contract (see later), the fair value of the interest rate swap is zero: neither the receiver nor the payer has a privileged position in the contract and thus none of them needs to pay a premium.

The value of r for which $\Pi_{\mathrm{irs}}(0) = 0$ is called the **(fair) swap rate** of the interest rate swap.

Theorem 2. *The swap rate of an interest rate swap stipulated at time $t = 0$ and with maturity T is given by*

$$r_{\mathrm{swap}} = \frac{\sum_{i=1}^{n-1} B(0, T_{i+1}) F_\delta(0, T_i)}{\sum_{i=1}^{n-1} B(0, T_{i+1})}. \tag{10}$$

$$F_\delta(t, T_i) = \frac{B(t,T_i) - B(t,T_{i+1})}{\delta B(t, T_{i+1})}$$

$D(t) = e^{-\int_0^t r(s)\,ds}$

Proof. We show below that, for all $T > 0$ and $\delta > 0$, the following identity holds:

$$\widetilde{\mathbb{E}}[D(T+\delta)F_\delta(T,T)] = B(0,T+\delta)F_\delta(0,T). \tag{11}$$

$R_\delta(T)$

Using (11) in (9) we obtain

$$\begin{aligned}\Pi_{\mathrm{irs}}(0) &= \delta\sum_{i=1}^{n-1}\widetilde{\mathbb{E}}[F_\delta(T_i,T_i)D(T_i+\delta)] - \delta r\sum_{i=1}^{n-1}\widetilde{\mathbb{E}}[D(T_{i+1})] \\ &= \delta\left(\sum_{i=1}^{n-1}B(0,T_{i+1})F_\delta(0,T_i) - r\sum_{i=1}^{n-1}B(0,T_{i+1})\right),\end{aligned} \tag{12}$$

hence $\Pi_{\mathrm{irs}}(0) = 0$ if and only if $r = r_{\mathrm{swap}}$. It remains to prove (11). As $B(t,T) = \widetilde{\mathbb{E}}[D(T)/D(t)|\mathcal{F}_W(t)]$, we have

$$\begin{aligned}\widetilde{\mathbb{E}}[D(T+\delta)F_\delta(T,T)] &= \widetilde{\mathbb{E}}[D(T+\delta)(\frac{1-B(T,T+\delta)}{\delta B(T,T+\delta)})] \\ &= \frac{1}{\delta}\widetilde{\mathbb{E}}[D(T+\delta)B(T,T+\delta)^{-1}] - \frac{1}{\delta}\widetilde{\mathbb{E}}[D(T+\delta)] \\ &= \frac{1}{\delta}\widetilde{\mathbb{E}}[\widetilde{\mathbb{E}}[\frac{D(T+\delta)}{D(T)}\frac{D(T)}{B(T,T+\delta)}|\mathcal{F}_W(T)]] - \frac{1}{\delta}B(0,T+\delta) \\ &= \frac{1}{\delta}\widetilde{\mathbb{E}}[\frac{D(T)}{B(T,T+\delta)}B(T,T+\delta)] - \frac{1}{\delta}B(0,T+\delta) \\ &= \frac{1}{\delta}\frac{B(0,T)-B(0,T+\delta)}{B(0,T+\delta)}B(0,T+\delta) = B(0,T+\delta)F_\delta(0,T).\end{aligned}$$

□

Remark. Note carefully that all quantities in the right hand side of (10) are known at time $t = 0$, hence the swap rate is fixed by information available at the time when the interest rate swap is stipulated.

Caps and Floors

An **interest rate cap** is a contract that caps (i.e., put a maximum limit on) the spot rate. More precisely, consider, as before, a uniform partition $0 = T_0 < T_1 < \cdots < T_n = T$ of the interval $[0,T]$ with $T_i - T_{i-1} = \delta$, for all $i = 1,\ldots,n$.

Let $R_\delta(T_i) = F_\delta(T_i,T_i)$ be the simply compounded spot rate in the interval $[T_i,T_{i+1}]$.

An interest rate cap with strike rate r and notional amount $N = 1$ pays to its owner the amount $(R_\delta(T_i)\delta - r\delta)_+$ at time T_{i+1}, $i = 1,\ldots,n-1$.

Hence the spot rate for the owner of the interest rate cap is no higher than r: any excess to the strike rate is paid by the seller of the interest rate cap.

Similarly, an **interest rate floor** put a minimum on the spot rate and pays to its owner the amount $(r\delta - R_\delta(T_i)\delta)_+$ at every time T_{i+1}, $i = 1, \dots, n-1$.

The risk-neutral price of the interest rate cap/floor at time $t = 0$ is given by

$$\Pi_{\rm cap}(0) = \delta \sum_{i=1}^{n-1} \widetilde{\mathbb{E}}[(R_\delta(T_i) - r)_+ D(T_{i+1})], \tag{13}$$

$$\Pi_{\rm floor}(0) = \delta \sum_{i=1}^{n-1} \widetilde{\mathbb{E}}[(r - R_\delta(T_i))_+ D(T_{i+1})]. \tag{14}$$

As $(R_\delta(T_i) - r)_+ - (r - R_\delta(T_i))_+ = (R_\delta(T_i) - r)$, the **cap-floor parity** identity holds:

$$\Pi_{\rm cap}(0) - \Pi_{\rm floor}(0) = \Pi_{\rm irs}(0).$$

In particular if the strike rate coincides with the swap rate then the cap and the floor have the same initial price. An interest rate cap (resp. floor) on one time period (i.e., $n = 1$) is called a **caplet** (resp. **floorlet**).