Lecture 21

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Financial derivatives and PDE's
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The HJM approach to ZCB pricing

Next we present a different approach for the evaluation of ZCB's due to Heath, Jarrow and Morton (HJM model).

The cornerstone of this approach is to use the forward rate of the market, instead of the spot rate, as the fundamental parameter to express the price of the bond.

The starting point in the HJM approach is to assume that $\{F(t,T), t \in [0,T], T \in [0,S]\}$ is given by the diffusion process

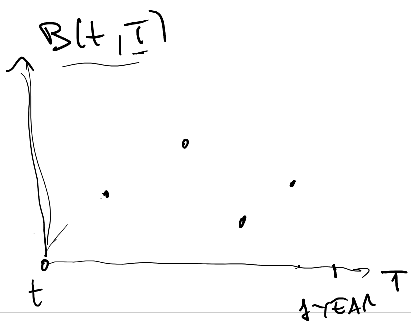
$$dF(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t), \quad t \in [0,T], \ T \in [0,S]. \tag{1}$$

$$dr(t) = a(b - r(t))dt + c\sqrt{r(t)}\,d\widetilde{W}(t)$$

where $\{\alpha(t,T), t \in [0,T], T \in [0,S]\}$, $\{\sigma(t,T), t \in [0,T], T \in [0,S]\} \in \mathcal{C}^0[\mathcal{F}_W(t)]$; in the applications they are often assumed to be deterministic functions of (t,T).

Note carefully that, as opposed to the spot rate dynamics in the classical approach, the forward rate dynamics in the HJM approach is given in terms of the *physical* (real-world) probability and not *a priori* in the risk-neutral probability.

Specifying the dynamics (1) for the forward rate in the HJM approach corresponds to the assumption made in options pricing theory that the underlying stock follows a (generalized) geometric Brownian motion.



$$F(t,T) = -\partial_T \log B(t,T)$$

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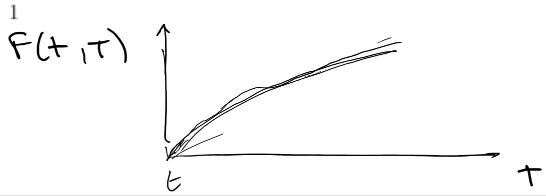

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Theorem 1. *When the forward rate of the ZCB market is given by* (1)*, the value of the zero-coupon bond is the diffusion process* $\{B(t,T), t \in [0,T], T \in [0,S]\}$ *given by*

$$dB(t,T) = B(t,T)[r(t) - \overline{\alpha}(t,T) + \frac{1}{2}\overline{\sigma}(t,T)^2]dt - \overline{\sigma}(t,T)B(t,T)\,dW(t), \tag{2}$$

where $r(t)$ *is the instantaneous spot rate and*

$$\overline{\alpha}(t,T) = \int_t^T \alpha(t,v)\,dv, \quad \overline{\sigma}(t,T) = \int_t^T \sigma(t,v)\,dv. \tag{3}$$

$$B(t,T) = e^{-\int_t^T F(t,v)\,dv} = e^{X(t)}$$

Proof. Let $X(t) = -\int_t^T F(t,v)\,dv$. By Itô's formula,

$$dB(t,T) = B(t,T)(dX(t) + \frac{1}{2}dX(t)dX(t)). \tag{4}$$

$$dB = B\left(dX + \frac{1}{2}dXdX\right)$$

Moreover

$$\begin{aligned} dX(t) &= F(t,t)\,dt - \int_t^T dF(t,v)\,dv \\ &= r(t)\,dt - \int_t^T (\alpha(t,v)\,dt + \sigma(t,v)\,dW(t))\,dv \\ &= r(t)\,dt - \overline{\alpha}(t,T)\,dt - \overline{\sigma}(t,T)\,dW(t). \end{aligned}$$

$$dX\,dX = \overline{\sigma}^2(t,T)\,dt$$

Replacing in (4) the claim follows. □

We now establish a condition which ensures that investing on ZCB's entails no arbitrage.

This can be achieved by showing that there exists a probability measure $\widetilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ such that the discounted value of the ZCB is a martingale for *all* maturities $T \in [0,S]$.

Recall that $\{\overline{\alpha}(t,T), t \in [0,T], T \in [0,S]\}$ and $\overline{\sigma}(t,T)t \in [0,T], T \in [0,S]\}$ are given by (3).

Theorem 2. *Let the forward rate process* $\{F(t,T), t \in [0,T], T \in [0,S]\}$ *be given by* (1) *such that* $\sigma(t,T) > 0$ *a.s. for all* $0 \le t \le T \le S$*. Assume that there exists a stochastic process* $\{\theta(t)\}_{t\in[0,S]} \in \mathcal{C}^0[\mathcal{F}_W(t)]$ *(the* **market price of risk***), independent of* $T \in [0,S]$*, such that*

$$\alpha(t,T) = \theta(t)\sigma(t,T) + \sigma(t,T)\overline{\sigma}(t,T), \text{ for all } 0 \le t \le T \le S \tag{5}$$

$$\theta(t) = \frac{\alpha(t,T) - \sigma(t,T)\overline{\sigma}(t,T)}{\sigma(t,T)}$$

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and such that the stochastic process $\{Z(t)\}_{t\in[0,S]}$ *given by*

$$Z(t) = \exp\left(-\int_0^t \theta(s)\,dW(s) - \frac{1}{2}\int_0^t \theta(s)^2\,ds\right)$$

is a $\mathbb{P}$*-martingale (e.g.,* $\{\theta(t)\}_{t\in[0,S]}$ *satisfies the Novikov condition). Let* $\widetilde{\mathbb{P}}$ *be the probability measure equivalent to* $\mathbb{P}$ *given by* $\widetilde{\mathbb{P}}(A) = \mathbb{E}[Z(\hat{S})\mathbb{I}_A]$*, for all* $A \in \mathcal{F}$*, and denote by* $\{\widetilde{W}(t)\}_{t\in[0,S]}$ *the* $\widetilde{\mathbb{P}}$*-Brownian motion given by* $d\widetilde{W}(t) = dW(t) + \theta(t)\,dt$*. Then the following holds:*

(i) The forward rate satisfies

$$dF(t,T) = \sigma(t,T)\overline{\sigma}(t,T)\,dt + \sigma(t,T)\,d\widetilde{W}(t). \tag{6}$$

(ii) The discounted price of the ZCB with maturity $T \in [0,S]$ satisfies

$$dB^*(t,T) = -\overline{\sigma}(t,T)B^*(t,T)\,d\widetilde{W}(t). \tag{7}$$

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In particular for any given $T \in [0,S]$, the discounted value process $\{B^*(t,T)\}_{t\in[0,T]}$ is a $\widetilde{\mathbb{P}}$-martingale relative to $\{\mathcal{F}_W(t)\}_{t\geq 0}$, and so any self-financing portfolio that consists of ZCB's with different maturities is not an arbitrage.

(iii) The value of the ZCB satisfies the risk-neutral pricing formula.

$B(t,T) = \widetilde{\mathbb{E}}\left[\frac{D(T)}{D(t)} \,\Big|\, \mathcal{F}_W(t)\right]$

The probability measure $\widetilde{\mathbb{P}}$ is called the **risk neutral probability measure** *of the ZCB market.*

Proof. (6) follows at once by replacing (5) into (1) and using $d\widetilde{W}(t) = dW(t) + \theta(t)\,dt$. In order to prove (7) we first show that $\theta(t)$ can be rewritten as

$$\theta(t) = \frac{\overline{\alpha}(t,T)}{\overline{\sigma}(t,T)} - \frac{1}{2}\overline{\sigma}(t,T). \tag{8}$$

Indeed, integrating (5) with respect to T and using $\sigma(t,T) = \partial_T\overline{\sigma}(t,T)$ we obtain

$$\begin{aligned}
\overline{\alpha}(t,T) &= \theta(t)\overline{\sigma}(t,T) + \int_t^T \sigma(t,v)\overline{\sigma}(t,v)\,dv \\
&= \theta(t)\overline{\sigma}(t,T) + \frac{1}{2}\int_t^T \partial_v(\overline{\sigma}(t,v)^2)\,dv \\
&= \theta(t)\overline{\sigma}(t,T) + \frac{1}{2}\overline{\sigma}(t,T)^2,
\end{aligned}$$

which gives (8). Next, by (2) and Itô's product rule,

$$\begin{aligned}
d(D(t)B(t,T)) &= D(t)B(t,T)(r(t) - \overline{\alpha}(t,T) + \frac{1}{2}\overline{\sigma}(t,T)^2)\,dt \\
&\quad - D(t)B(t,T)\overline{\sigma}(t,T)\,dW(t) - B(t,T)D(t)r(t)\,dt \\
&= B^*(t,T)[(\frac{1}{2}\overline{\sigma}(t,T)^2 - \overline{\alpha}(t,T))\,dt - \overline{\sigma}(t,T)\,dW(t)] \\
&= -B^*(t,T)\overline{\sigma}(t,T)\,d\widetilde{W}(t),
\end{aligned}$$

where in the last step we use (8). Thus $\{B^*(t,T)\}_{t\in[0,T]}$ is a $\widetilde{\mathbb{P}}$-martingale relative to $\{\mathcal{F}_W(t)\}_{t\geq 0}$. In particular

$$\widetilde{\mathbb{E}}[D(T)\overbrace{B(T,T)}^{1}|\mathcal{F}_W(t)] = D(t)B(t,T).$$

$\Rightarrow B(t,T) = \widetilde{\mathbb{E}}\left[\frac{D(T)}{D(t)} \,\Big|\, \mathcal{F}_W(t)\right]$

As $B(T,T) = 1$, the previous equation is equivalent to the risk-neutral pricing formula. $\square$

Example: The CIR model revisited

As a way of example we show how to re-formulate the CIR model in the HJM approach. Recall that in the CIR model the spot rate is given by

$$dr(t) = a(b - r(t))\, dt + c\sqrt{r(t)}\, d\widetilde{W}(t). \tag{9}$$

We have seen before that the price of the zero-coupon bond with face value 1 and maturity $T > 0$ in the CIR model is given by $B(t, T) = v(t, r(t))$, where $v(t, x) = \exp(-xC(T - t) - A(T - t))$ and where the deterministic functions $A(\tau), C(\tau)$ satisfy

$\tau = T - t$

$$C'(\tau) = 1 - aC(\tau) - \frac{c^2}{2}C(\tau)^2, \quad A'(\tau) = abC(\tau) \tag{10a}$$

$$C(0) = 0, \quad A(0) = 0. \tag{10b}$$

The instantaneous forward rate satisfies

$$F(t, T) = -\partial_T \log B(t, T) = r(t)C'(T - t) + A'(T - t).$$

Hence, using (9),

$$\begin{aligned}
dF(t, T) &= C'(T - t)dr(t) - r(t)C''(T - t)dt - A''(T - t)dt \\
&= [a(b - r(t))\, dt + c\sqrt{r(t)}\, d\widetilde{W}(t)] - r(t)C''(T - t)\, dt - A''(T - t)\, dt \\
&= [a(b - r(t))C'(T - t) - r(t)C''(T - t) - A''(T - t)]\, dt + C'(T - t)c\sqrt{r(t)}\, d\widetilde{W}(t).
\end{aligned}$$

Comparing this result with (6) we are led to set

$$\sigma(t, T) = C'(T - t)c\sqrt{r(t)}, \tag{11}$$

$$\sigma(t, T)\overline{\sigma}(t, T) = [a(b - r(t))C'(T - t) - r(t)C''(T - t) - A''(T - t)]. \tag{12}$$

As $\overline{\sigma}(t, T) = \int_t^T \sigma(t, v)\, dv$, we obtain

$$\begin{aligned}
&a(b - r(t))C'(T - t) - r(t)C''(T - t) - A''(T - t) \\
&= C'(T - t)c\sqrt{r(t)} \int_t^T C'(v - t)c\sqrt{r(t)}\, dv \\
&= c^2 r(t)C'(T - t)C(T - t).
\end{aligned}$$

$\sigma(t,T)\,\overline{\sigma}(t,T) = c^2 r(t)\, C'(T-t)\, C(T-t)$

$\sigma(t,T) = C'(T-t)\, c\sqrt{r(t)}$

The latter identity holds for all possible value of $r(t)$ if and only if the following two equations are satisfied:

$$\underbrace{abC' - A'' = 0}, \quad \underbrace{-aC' - C'' = -c^2 C'C}. \tag{13}$$

Differentiating (10a) we find that (13) are indeed satisfied.

Note that the HJM approach gives the same result (i.e., the same pricing function for ZCB's) as the classical approach with the CIR model if we assume that the forward rate is given by (1) with $\alpha(t,T)$ given by (5) (where $\theta(t)$ is arbitrary, typically chosen to be constant) and $\sigma(t,T)$ is given by (11).

The advantage of the HJM approach is that the model for the forward rate is expressed in the physical probability (as opposed to (9)) and thus the parameters of this model can be calibrated using real market data.

Exercise 1. *Assume that the spot interest rate in the risk-neutral probability is given by the Ho-Lee model:*

$$dr(t) = a(t)dt + c\,d\widetilde{W}(t), \iff r(t) = r(0) + \int_0^t a(s)\,ds + c\,\widetilde{W}(t)$$

where $c > 0$ is a constant and $a(t)$ is a deterministic function of time. Derive the risk-neutral price $B(t,T)$ of the ZCB with face value 1 and maturity T. [Use the HJM method to derive the dynamics of the instantaneous forward rate $F(t,T)$ in the physical probability.]

$$r(t) \in N\left(r(0) + \int_0^t a(s)\,ds,\ c^2 t\right)$$

SOLUTION:

$$B(t,T) = \widetilde{\mathbb{E}}\left[e^{-\int_t^T r(s)ds} \,\middle|\, \mathcal{F}_W(t) \right] = v(t, r(t))$$

$$d\left(D(t)\,v(t,r(t))\right) = (dD)\,v + D\,dv + \underbrace{dD\,dv}_{0}$$

$$= \left\{ D(t) = e^{-\int_0^t r(s)ds} \Rightarrow dD = -r(t)D(t)dt \right\}$$

$$dv = \partial_t v\,dt + \partial_x v\,dr(t) + \frac{1}{2}\partial_x^2 v\,dr(t)dr(t)$$

$$= \underbrace{(\ldots)_1}\,dt + (\ldots)_2\,d\widetilde{W}(t)$$

$$(\ldots)_1 = 0 \iff \partial_t v + a(t)\partial_x v + \frac{1}{2}c^2\partial_x^2 v = x v \qquad v(T,x) = 1$$

$$v(t,x) = e^{-x\tau(t) - A(t)}$$

$$v\left(-x\tau'(t) - A'(t) - a(t)\tau(t) + \frac{1}{2}c^2\tau(t)^2 - x\right) = 0$$

$$\tau'(t) = -1,\ \tau(T) = 0 \Rightarrow \tau(t) = T - t$$

$$A'(t) = -a(t)\tau(t) + \frac{1}{2}c^2\tau(t)^2$$

$$\Rightarrow -\underset{\overset{\parallel}{0}}{A(T)} + A(t) = +\int_t^T a(s)\tau(s)ds - \frac{1}{2}c^2\int_t^T \underset{T-s}{\tau(s)}^2 ds$$

$$A(t) = \frac{1}{2}c^2\frac{(T-t)^3}{3} + \int_t^T a(s)(T-s)\,ds$$

$$v(t,x) = e^{-x(T-t) - \frac{1}{2}c^2\frac{(T-t)^3}{3} - \int_t^T \theta(s)(T-s)\,ds}$$

$$F(t,T) = -\partial_T \log B(t,T) = -\partial_T \log v(t, r(t))$$

$$= r(t)\,\partial_T \tau + \partial_T A \qquad 0 \quad (\tau' = -1)$$

$$dF(t,T) = \underbrace{(\partial_T \tau)}_{1 \ (\tau(t) = T-t)}\, dr(t) + r(t)\,\cancel{\partial_T(\tau'(t))} + \underbrace{\partial_T A'(t)}$$

$$= c^2(T-t)\,dt + c\,d\tilde{W}(t)$$

$$= \sigma(t,T)\,\tilde{\sigma}(t,T)\,dt + \sigma(t,T)\,d\tilde{W}(t)$$

$$\Rightarrow \sigma(t,T) = c \qquad \tilde{\sigma}(t,T) = c(T-t) = \int_t^T \sigma(t,u)\,du$$

$$\alpha(t,T) = c\,\theta(t) + c^2(T-t)$$

$$dF(t,T) = c\,(\theta(t) + c(T-t))\,dt + c\,dW(t)$$

$$F(t,T) \in N\left(c\int_0^t \theta(s)\,ds + c^2(T-t),\ c^2 t\right)$$