Financial derivatives and PDE's
Lecture 25

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Multi-assets options

Multi-asset options are options on several underlying assets. Notable examples include rainbow options, basket options and quanto options.

In the following we discuss options on two stocks in a **2+1 dimensional Black-Scholes market**, i.e., a market with constant parameters. It follows that

$$dS_1(t) = \mu_1 S_1(t)dt + \sigma_{11}S_1(t)dW_1(t) + \sigma_{12}S_1(t)dW_2(t) \tag{1a}$$
$$dS_2(t) = \mu_2 S_2(t)dt + \sigma_{21}S_2(t)dW_1(t) + \sigma_{22}S_2(t)dW_2(t), \tag{1b}$$

where the volatility matrix

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

is invertible (so that the market is complete).

Integrating (1) we obtain that $(S_1(t), S_2(t))$ is given by the **2-dimensional geometric Brownian motion**:

$$S_1(t) = S_1(0)e^{(\mu_1 - \frac{1}{2}(\sigma_{11}^2+\sigma_{12}^2))t + \sigma_{11}W_1(t) + \sigma_{12}W_2(t)}, \tag{2a}$$
$$S_2(t) = S_2(0)e^{(\mu_2 - \frac{1}{2}(\sigma_{21}^2+\sigma_{22}^2))t + \sigma_{21}W_1(t) + \sigma_{22}W_2(t)}, \tag{2b}$$

or, more concisely,

$$S_j(t) = S_j(0)e^{(\mu_j - \frac{|\sigma_j|^2}{2})t + \sigma_j \cdot W(t)},$$

where $\sigma_j = (\sigma_{j1}, \sigma_{j2})$, $j = 1,2$, $W(t) = (W_1(t), W_2(t))$ and $\cdot$ denotes the standard scalar product of vectors.

Theorem 1. *The random variables $S_1(t), S_2(t)$ have the joint density*

$$f_{S_1(t),S_2(t)}(x,y) = \frac{e^{-\frac{1}{2t}\left(\log\frac{x}{S(0)} - \alpha_1 t \quad \log\frac{y}{S(0)} - \alpha_2 t\right)(\sigma\sigma^T)^{-1}\begin{pmatrix}\log\frac{x}{S(0)} - \alpha_1 t \\ \log\frac{y}{S(0)} - \alpha_2 t\end{pmatrix}}}{txy\sqrt{(2\pi)^2\det(\sigma\sigma^T)}}, \tag{3}$$

where $\alpha_j = \mu_j - \frac{|\sigma_j|^2}{2}$, $j = 1,2$. Moreover $\log S_1(t), \log S_2(t)$ are jointly normally distributed with mean $m = (\log S_1(0) + \alpha_1 t, \log S_2(0) + \alpha_2 t)$ and covariant matrix $C = t\sigma\sigma^T$.

Proof. Letting $X_i = W_i(t)/\sqrt{t} \in \mathcal{N}(0,1)$, we write the stock prices as

$$S_1(t) = S_1(0)e^{\alpha_1 t + Y_1}, \quad S_2(t) = S_2(0)e^{\alpha_2 t + Y_2},$$

where

$$Y_1 = \sigma_{11}\sqrt{t}X_1 + \sigma_{12}\sqrt{t}X_2, \quad Y_2 = \sigma_{21}\sqrt{t}X_1 + \sigma_{22}\sqrt{t}X_2.$$

It follows that Y_1, Y_2 are jointly normally distributed with zero mean and covariant matrix $C = t\sigma\sigma^T$, which proves the second statement in the theorem. To compute the joint density of the stock prices, we notice that

$$S_1(t) \le x \Leftrightarrow Y_1 \le \log\left(\frac{x}{S_1(0)}\right) - \alpha_1 t, \quad S_2(t) \le y \Leftrightarrow Y_2 \le \log\left(\frac{y}{S_2(0)}\right) - \alpha_2 t,$$

hence

$$F_{S_1(t),S_2(t)}(x,y) = F_{Y_1,Y_2}(\log\frac{x}{S_1(0)} - \alpha_1 t, \log\frac{y}{S_2(0)} - \alpha_2 t).$$

Hence

$$f_{S_1(t),S_2(t)}(x,y) = \partial^2_{xy}F_{S_1(t),S_2(t)}(x,y) = \frac{1}{xy}f_{Y_1,Y_2}(\log\frac{x}{S_1(0)} - \alpha_1 t, \log\frac{y}{S_2(0)} - \alpha_2 t).$$

Using the joint normal density of Y_1, Y_2 completes the proof. $\square$

Exercise 1. *Show that the process* (1) *is equivalent, in distribution, to the process*

$$dS_i(t) = \mu_i S_i(t)\,dt + \overline{\sigma}_i S_i(t) dW_i^{(\rho)}(t), \quad i = 1, 2, \tag{4}$$

where

$$\overline{\sigma}_i = \sqrt{\sigma_{i1}^2 + \sigma_{i2}^2}, \quad \rho = \frac{\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22}}{\sqrt{(\sigma_{11}^2 + \sigma_{12}^2)(\sigma_{21}^2 + \sigma_{22}^2)}} \in [-1, 1] \tag{5}$$

and where $W_1^{(\rho)}(t)$, $W_2^{(\rho)}(t)$ *are correlated Brownian motions with correlation* ρ, *i.e.,*

$$dW_1^{(\rho)}(t) dW_2^{(\rho)}(t) = \rho\,dt.$$

Now let $r(t) = r$ be the constant interest rate of the money market. The solution of the market price of risk equations can be written as

$$\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \sigma^{-1} \begin{pmatrix} \mu_1 - r \\ \mu_2 - r \end{pmatrix} = \frac{1}{\det \sigma} \begin{pmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{21} & \sigma_{11} \end{pmatrix} \begin{pmatrix} \mu_1 - r \\ \mu_2 - r \end{pmatrix},$$

that is

$$\theta_1 = \frac{1}{\det \sigma}[\sigma_{22}(\mu_1 - r) - \sigma_{12}(\mu_2 - r)], \quad \theta_2 = \frac{1}{\det \sigma}[-\sigma_{21}(\mu_1 - r) + \sigma_{11}(\mu_2 - r)].$$

Replacing $dW_i(t) = d\widetilde{W}_i(t) - \theta_i\, dt$ into (1) we find

$$dS_1(t) = r\, S_1(t)dt + \sigma_{11}S_1(t)d\widetilde{W}_1(t) + \sigma_{12}S_1(t)d\widetilde{W}_2(t), \tag{6a}$$

$$dS_2(t) = r\, S_2(t)dt + \sigma_{21}S_2(t)d\widetilde{W}_1(t) + \sigma_{22}S_2(t)d\widetilde{W}_2(t). \tag{6b}$$

Note that the discounted price of both stocks is a martingale in the risk-neutral probability measure, as expected.

Moreover the system (6) can be integrated to give

$$S_j(t) = S_j(0)e^{(r - \frac{|\sigma_j|^2}{2})t + \sigma_j \cdot \widetilde{W}(t)}, \tag{7}$$

where $\widetilde{W}(t) = (\widetilde{W}_1(t), \widetilde{W}_2(t))$. As $\widetilde{W}_1(t), \widetilde{W}_2(t)$ are independent $\widetilde{\mathbb{P}}$-Brownian motions, the joint distribution of the stock prices in the risk-neutral probability measure is given by (3) where now

$$\alpha_i = r - \frac{|\sigma_j|^2}{2}, \quad i = 1, 2.$$

Next consider a standard European style derivative on the two stocks with pay-off $Y = g(S_1(T), S_2(T))$. The risk-neutral price of the derivative is

$$\Pi_Y(t) = e^{-r(T-t)}\widetilde{\mathbb{E}}[g(S_1(T), S_2(T))|\mathcal{F}_W(t)]. \tag{8}$$

By the Markov property for systems of stochastic differential equations, there exists a function $v_g : [0,T] \times (0,\infty)^2 \to (0,\infty)$ such that

$$\Pi_Y(t) = v_g(t, S_1(t), S_2(t)). \tag{9}$$

As in the case of options on one single stock, the pricing function can be computed in two ways: using the joint probability density of the stocks or by solving a PDE.

Black-Scholes price for options on two stocks

We show first how to compute the function v_g in (9) using the joint probability density of $S_1(t), S_2(t)$ derived in Theorem 1. We argue as in the one-dimensional case.

By (7) we have

$$S_i(T) = S_j(t) e^{(r - \frac{|\sigma_j|^2}{2})\tau + \sigma_j \cdot (\widetilde{W}(T) - \widetilde{W}(t))}, \quad \tau = T - t.$$

Replacing into (8) we obtain

$$\Pi_Y(t) = e^{-r\tau} \widetilde{\mathbb{E}}[g(S_1(t) e^{(r - \frac{|\sigma_1|^2}{2})\tau + \sigma_1 \cdot (\widetilde{W}(T) - \widetilde{W}(t))}, S_2(t) e^{(r - \frac{|\sigma_2|^2}{2})\tau + \sigma_2 \cdot (\widetilde{W}(T) - \widetilde{W}(t))}) | \mathcal{F}_W(t)].$$

As $(S_1(t), S_2(t))$ is measurable with respect to $\mathcal{F}_W(t)$ and $\widetilde{W}(T) - \widetilde{W}(t)$ is independent of $\mathcal{F}_W(t)$, the Independence Lemma gives

$$\Pi_Y(t) = v_g(t, S_1(t), S_2(t)),$$

where

$$v_g(t,x,y) = e^{-r\tau} \widetilde{\mathbb{E}}[g(x e^{(r - \frac{|\sigma_1|^2}{2})\tau + \sigma_1 \cdot (\widetilde{W}(T) - \widetilde{W}(t))}, y e^{(r - \frac{|\sigma_2|^2}{2})\tau + \sigma_2 \cdot (\widetilde{W}(T) - \widetilde{W}(t))})].$$

To compute the expectation in the right hand side of the latter equation we use that the random variables

$$Y_1 = \sigma_1 \cdot (\widetilde{W}(T) - \widetilde{W}(t)), \quad Y_2 = \sigma_2 \cdot (\widetilde{W}(T) - \widetilde{W}(t))$$

are jointly normally distributed with zero mean and covariance matrix $C = \tau \sigma \sigma^T$. Hence

$$
v_g(t,x,y) = e^{-r\tau}\int_{\mathbb{R}}\int_{\mathbb{R}} g(xe^{(r-\frac{|\sigma_1|^2}{2})\tau+\sqrt{\tau}\xi}, ye^{(r-\frac{|\sigma_2|^2}{2})\tau+\sqrt{\tau}\eta}) \frac{\exp\left(-\frac{1}{2}\begin{pmatrix}\xi & \eta\end{pmatrix}(\sigma\sigma^T)^{-1}\begin{pmatrix}\xi \\ \eta\end{pmatrix}\right)}{2\pi\sqrt{\det(\sigma\sigma^T)}} d\xi\, d\eta. \tag{10}
$$

Definition 1. *The stochastic process* $\{\Pi_Y(t)\}_{t\in[0,T]}$ *given by* (9)-(10)*, is called the* **Black-Scholes price** *of the standard 2-stocks European derivative with pay-off* $Y = g(S_1(T), S_2(T))$ *and time of maturity* $T > 0$*.*

Black-Scholes PDE for options on two stocks

Next we show how to derive the pricing function v_g by solving a PDE.

Theorem 2. *Let* v_g *be the (unique) strong solution to the terminal value problem*

$$
\begin{aligned}
&\partial_t v_g + r(x\partial_x v_g + y\partial_y v_g) + \frac{1}{2}(\sigma_{11}^2+\sigma_{12}^2)x^2\partial_x^2 v_g + \frac{1}{2}(\sigma_{21}^2+\sigma_{22}^2)y^2\partial_y^2 v_g \\
&\quad + (\sigma_{11}\sigma_{21}+\sigma_{12}\sigma_{22})xy\partial_{xy}v_g = r\, v_g, \quad t\in(0,T), \quad x,y>0, && \text{(11a)}\\
&v_g(T,x,y) = g(x,y), \quad x,y>0. && \text{(11b)}
\end{aligned}
$$

Then (9) *holds. The PDE in* (11) *is called the* **2-dimensional Black-Scholes PDE***.*

Proof. By Itô's formula in two dimensions,

$$
\begin{aligned}
d(e^{-rt}v_g) =& e^{-rt}\Big(-rv_g\,dt + \partial_t v_g\,dt + \partial_x v_g\,dS_1(t) + \partial_y v_g\,dS_2(t) \\
&+ \partial_{xy}^2 v_g\,dS_1(t)dS_2(t) + \frac{1}{2}\partial_x^2 v_g\,dS_1(t)\,dS_1(t) + \frac{1}{2}\partial_y^2 v_g\,dS_2(t)\,dS_2(t)\Big).
\end{aligned}
$$

Moreover, using (6),

$$
\begin{aligned}
dS_1(t)dS_1(t) &= (\sigma_{11}^2+\sigma_{12}^2)S_1(t)^2\,dt \\
dS_2(t)dS_2(t) &= (\sigma_{21}^2+\sigma_{22}^2)S_2(t)^2\,dt \\
dS_1(t)dS_2(t) &= (\sigma_{11}\sigma_{21}+\sigma_{12}\sigma_{22})S_1(t)S_2(t)\,dt.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
d(e^{-rt}v_g(t,S_1(t),S_2(t))) = \alpha(t)\,dt &+ e^{-rt}S_1(t)\partial_x v_g(t,S_1(t),S_2(t))\,(\sigma_{11}\,d\widetilde{W}_1(t) + \sigma_{12}d\widetilde{W}_2(t)) \\
&+ e^{-rt}S_2(t)\partial_y v_g(t,S_1(t),S_2(t))\,(\sigma_{21}\,d\widetilde{W}_1(t) + \sigma_{22}d\widetilde{W}_2(t))
\end{aligned}
$$

where the drift term is

$$\begin{aligned}\alpha(t) =& e^{-rt}\big(- rv_g + \partial_t v_g + r(x\partial_x v_g + y\partial_y v_g) \\ &+ \frac{1}{2}(\sigma_{11}^2 + \sigma_{12}^2)x^2\partial_x^2 v_g + \frac{1}{2}(\sigma_{21}^2 + \sigma_{22}^2)y^2\partial_y^2 v_g \\ &+ (\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22})xy\partial_{xy}v_g\big)(t, S_1(t), S_2(t)) = 0,\end{aligned}$$

due to v_g solving (11). It follows that the stochastic process $\{e^{-rt}v_g(t, S_1(t), S_2(t))\}_{t\in[0,T]}$ is a $\widetilde{\mathbb{P}}$-martingale relative to $\{\mathcal{F}_W(t)\}_{t\geq 0}$, hence, using the terminal condition $v_g(T) = g$, we have

$$e^{-rT}\widetilde{\mathbb{E}}[g(S_1(T), S_2(T))|\mathcal{F}_W(t)] = e^{-rt}v_g(t, S_1(t), S_2(t)), \quad t \in [0, T],$$

which proves (9). □

Hedging portfolio

Finally we derive the formulas for the hedging portfolio for standard 2-stocks European derivatives in Black-Scholes markets.

Theorem 3. *The numbers of shares $h_{S_1}(t), h_{S_2}(t)$ in the self-financing hedging portfolio for the European derivative with pay-off $Y = g(S_1(T), S_2(T))$ and maturity T are given by*

$$h_{S_1}(t) = \partial_x v_g(t, S_1(t), S_2(t)), \quad h_{S_2}(t) = \partial_y v_g(t, S_1(t), S_2(t)).$$

Proof. The discounted value of the derivative satisfies $d\Pi_Y^*(t) = \Delta_1(t)d\widetilde{W}_1(t) + \Delta_2(t)d\widetilde{W}_2(t)$, where

$$\begin{aligned}\Delta_1(t) &= e^{-rt}(S_1(t)\sigma_{11}\partial_x v_g + S_2(t)\sigma_{21}\partial_y v_g)(t, S_1(t), S_2(t)) \\ \Delta_2(t) &= e^{-rt}(S_1(t)\sigma_{12}\partial_x v_g + S_2(t)\sigma_{22}\partial_y v_g)(t, S_1(t), S_2(t))\end{aligned}$$

Letting $\Delta = (\Delta_1\ \Delta_2)^T$, we have $\Delta/e^{-rt} = \sigma^T Y$, where

$$Y = \begin{pmatrix} S_1(t)\partial_x v_g(t, S_1(t), S_2(t)) \\ S_2(t)\partial_y v_g(t, S_1(t), S_2(t)) \end{pmatrix}.$$

Hence the number of stock shares in the hedging portfolio is $h_{S_1}(t) = Y_1/S_1(t) = \partial_x v_g(t, S_1(t), S_2(t))$, $h_{S_2}(t) = Y_2/S_2(t) = \partial_y v_g(t, S_1(t), S_2(t))$, which concludes the proof of the theorem. □

An example of option on two stocks (outperformance option)

Let $K, T > 0$ and consider a standard European derivative with pay-off

$$Y = \left(\frac{S_1(T)}{S_2(T)} - K \right)_+$$

at time of maturity T. This is an example of outperformance option, i.e., an option that allows investors to benefit from the relative performance of two underslying assets.

Using (7), we can write the risk-neutral price of the derivative as

$$\begin{aligned}
\Pi_Y(t) &= e^{-r\tau}\widetilde{\mathbb{E}}\left[\left(\frac{S_1(T)}{S_2(T)} - K \right)_+ |\mathcal{F}_W(t) \right] \\
&= e^{-r\tau}\widetilde{\mathbb{E}}\left[\left(\frac{S_1(t)}{S_2(t)} e^{(\frac{|\sigma_2|^2}{2} - \frac{|\sigma_1|^2}{2})\tau + (\sigma_1 - \sigma_2)\cdot(\widetilde{W}(T) - \widetilde{W}(t))} - K \right)_+ |\mathcal{F}_W(t) \right].
\end{aligned}$$

Now we write

$$(\sigma_1 - \sigma_2)\cdot(\widetilde{W}(T) - \widetilde{W}(t)) = \sqrt{\tau}[(\sigma_{11} - \sigma_{21})G_1 + (\sigma_{12} - \sigma_{22})G_2] = \sqrt{\tau}(X_1 + X_2),$$

where $G_j = (\widetilde{W}_j(T) - \widetilde{W}_j(t))/\sqrt{\tau} \in \mathcal{N}(0,1)$, $j = 1,2$, hence $X_j \in \mathcal{N}(0, (\sigma_{1j} - \sigma_{2j})^2)$, $j = 1,2$.

In addition, X_1, X_2 are independent random variables, hence, as shown in Section **??**, $X_1 + X_2$ is normally distributed with zero mean and variance $(\sigma_{11} - \sigma_{21})^2 + (\sigma_{12} - \sigma_{22})^2 = |\sigma_1 - \sigma_2|^2$. It follows that

$$\Pi_Y(t) = e^{-r\tau}\widetilde{\mathbb{E}}\left[\left(\frac{S_1(t)}{S_2(t)} e^{(\frac{|\sigma_2|^2}{2} - \frac{|\sigma_1|^2}{2})\tau + \sqrt{\tau}|\sigma_1 - \sigma_2|G)} - K \right)_+ \right],$$

where $G \in \mathcal{N}(0,1)$. Hence, letting

$$\hat{r} = \frac{|\sigma_1 - \sigma_2|^2}{2} + \left(\frac{|\sigma_2|^2}{2} - \frac{|\sigma_1|^2}{2} \right)$$

and $a = e^{(\hat{r} - r)\tau}$, we have

$$\Pi_Y(t) = a e^{-\hat{r}\tau}\mathbb{E}\left[\left(\frac{S_1(t)}{S_2(t)} e^{(\hat{r} - \frac{|\sigma_1 - \sigma_2|^2}{2})\tau + \sqrt{\tau}|\sigma_1 - \sigma_2|G} - K \right)_+ \right]$$

Up to the multiplicative parameter a, this is the Black-Scholes price of a call on a stock with price $S_1(t)/S_2(t)$, volatility $|\sigma_1 - \sigma_2|$ and for an interest rate of the money market given by $\hat{r}$. Hence

$$\Pi_Y(t) = a\left(\frac{S_1(t)}{S_2(t)}\Phi(d_+) - Ke^{-\hat{r}\tau}\Phi(d_-)\right) := v(t, S_1(t), S_2(t)) = u(t, \frac{S_1(t)}{S_2(t)}) \tag{12}$$

where

$$d_\pm = \frac{\log\frac{S_1(t)}{KS_2(t)} + (\hat{r} \pm \frac{|\sigma_1-\sigma_2|^2}{2})\tau}{|\sigma_1 - \sigma_2|\sqrt{\tau}}.$$

As to the self-financing hedging portfolio, we have $h_{S_1}(t) = \partial_x v(t, S_1(t), S_2(t))$, $h_{S_2}(t) = \partial_y v(t, S_1(t), S_2(t))$, $j = 1, 2$. Therefore, recalling the delta function of the standard European call, we obtain

$$h_{S_1}(t) = \frac{a}{S_2(t)}\Phi(d_+), \quad h_{S_2}(t) = -\frac{aS_1(t)}{S_2(t)^2}\Phi(d_+).$$

The same result can be obtained by solving the terminal value problem (11).

Indeed, the form of the pay-off function of the derivative suggests to look for solutions of (11) of the form $v_g(t, x, y) = u(t, x/y)$. The function $u(t, z)$ satisfies a standard Black-Scholes PDE in 1+1 dimension, whose solution is given as in (12).